Relatively free nilpotent torsion-free groups and their Lie algebras*

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Abstract

Let K be a field of characteristic zero. For a torsion-free finitely generated nilpotent group G, we naturally associate four finite dimensional nilpotent Lie algebras over K, $\mathcal{L}_K(G)$, $\mathrm{grad}^{(\ell)}(\mathcal{L}_K(G))$, $\mathrm{grad}^{(g)}(\exp \mathcal{L}_K(G))$ and $L_K(G)$. Let $\mathfrak{T}_c$ be a torsion-free variety of nilpotent groups of class at most c. For a positive integer n, with $n \geq 2$, let $F_n(\mathfrak{T}_c)$ be the relatively free group of rank n in $\mathfrak{T}_c$. We prove that $\mathcal{L}_K(F_n(\mathfrak{T}_c))$ is relatively free in some variety of nilpotent Lie algebras, and $\mathcal{L}_K(F_n(\mathfrak{T}_c)) \cong L_K(F_n(\mathfrak{T}_c)) \cong \mathrm{grad}^{(\ell)}(\mathcal{L}_K(F_n(\mathfrak{T}_c))) \cong \mathrm{grad}^{(g)}(\exp \mathcal{L}_K(F_n(\mathfrak{T}_c)))$ as Lie algebras in a natural way. Furthermore $F_n(\mathfrak{T}_c)$ is a Magnus nilpotent group. Let G_1 and G_2 be torsion-free finitely generated nilpotent groups which are quasi-isometric. We prove that if G_1 and G_2 are relatively free of finite rank, then they are isomorphic. Let L be a relatively free nilpotent Lie algebra over $\mathbb{Q}$ of finite rank freely generated by a set X. Give on L the structure of a group R, say, by means of the Baker-Campbell-Hausdorff formula, and let H be the subgroup of R generated by the set X. We show that H is relatively free in some variety of nilpotent groups; freely generated by the set X, H is Magnus and $L \cong \mathcal{L}_{\mathbb{Q}}(H) \cong L_{\mathbb{Q}}(H)$ as Lie algebras. We extend the isomorphism between $\mathcal{L}_K$ and L_K to relatively free residually torsion-free nilpotent groups. We also give an example of a finitely generated Magnus nilpotent group G, not relatively free, such that $\mathcal{L}_{\mathbb{Q}}(G)$ is not isomorphic to $L_{\mathbb{Q}}(G)$ as Lie algebras.

Keywords and phrases: Varieties of groups, relatively free groups, varieties of Lie algebras, relatively free Lie algebras, Baker-Campbell-Hausdorff formula, Mal'cev completion, quasi-isometry.

Contents

1 Introduction and Notation

Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the ring of integers and the field of rational numbers, respectively. Furthermore we write $\mathbb{N}$ for the set of positive integers. For a group G and $i \in \mathbb{N}$, we write $\gamma_i(G)$ for the i-th term of the lower central series of G. Moreover we denote $G' = \gamma_2(G)$ i.e. the commutator subgroup of G. For elements a, b of G, we write $(a, b) = a^{-1}b^{-1}ab$, and for $c \geq 3$, and elements $a_1, \ldots, a_c$ of G, we define the left-normed group commutator $(a_1, \ldots, a_{c-1}, a_c) = ((a_1, \ldots, a_{c-1}), a_c)$. We call a group G, a *Magnus group* if each $\gamma_i(G)/\gamma_{i+1}(G)$ is torsion-free and $\cap_{i \geq 1}\gamma_i(G) = \{1\}$. For $n \in \mathbb{N}$, with $n \geq 2$, let F_n be a free group of rank n freely generated by the set $\{f_1, \ldots, f_n\}$. For a variety of groups $\mathfrak{G}$, let $\mathfrak{G}(F_n)$ denote the verbal subgroup of F_n corresponding to $\mathfrak{G}$. Also, let $F_n(\mathfrak{G}) = F_n/\mathfrak{G}(F_n)$: thus $F_n(\mathfrak{G})$ is a relatively free group of rank n in $\mathfrak{G}$ and it has a free generating set $\{x_1, \ldots, x_n\}$, where $x_i = f_i\mathfrak{G}(F_n)$, $i = 1, \ldots, n$. Note that the verbal subgroups of F_n are precisely the fully invariant subgroups of F_n (that is, the subgroups of F_n which are invariant under all group endomorphisms of F_n). The same property holds for verbal subgroups and fully invariant subgroups of relatively free groups. (For further information concerning relatively free groups and varieties of groups see [16].)

Let $\mathfrak{N}_c$ be the variety of nilpotent groups of class at most c, and let $\mathfrak{T}_c$ be a torsion-free subvariety of class at most c of $\mathfrak{N}_c$ (that is, its free groups of arbitrary rank are torsion-free).

Let K be a field of characteristic zero. We identify the prime subfield of K with $\mathbb{Q}$. By "Lie algebra "(resp. "Lie ring ") we mean a Lie algebra over K (resp. over $\mathbb{Z}$). For $n \in \mathbb{N}$, with $n \geq 2$, let L_n denote a free Lie algebra of rank n freely generated by the set $\{\ell_1, \ldots, \ell_n\}$. Let $\mathfrak{B}$ be a variety of Lie algebras and let $\mathfrak{B}(L_n)$ be the fully invariant ideal of L_n corresponding to $\mathfrak{B}$. Write $L_n(\mathfrak{V}) = L_n/\mathfrak{B}(L_n)$: thus $L_n(\mathfrak{B})$ is a relatively free Lie algebra of rank n in $\mathfrak{B}$ and it has a free generating set $\{\omega_1, \ldots, \omega_n\}$, where $\omega_i = \ell_i + \mathfrak{B}(L_n)$, $i = 1, \ldots, n$. As for groups the fully invariant ideals of L_n are precisely the verbal ideals of L_n. The same property holds for fully invariant ideals and verbal ideals of relatively free Lie algebras. (For further information concerning relatively free Lie algebras and varieties of Lie algebras see [1, Corollary 2.5, Chapter 14].) For $c \in \mathbb{N}$, with $c \geq 2$, we write $\mathfrak{M}_c$ for the variety of all Lie algebras defined by the identity $[\ell_1, \ldots, \ell_{c+1}] = 0$: the variety of all Lie algebras which are nilpotent of class at most c. Any variety of Lie algebras is assumed to be non-trivial. We write $\mathrm{var}(\mathfrak{X})$ for the variety generated by a set or a class $\mathfrak{X}$ of Lie algebras. (We use similar definition as for groups [16, page 18].)

Let $\widehat{L}_n$ be the completion of L_n with respect to the lower central series of L_n. (Recall that $\widehat{L}_n$ is identified with the complete (unrestricted) direct sum $\widehat{\oplus}_{m\geq 1} L_n^m$, and it has a natural Lie algebra structure. Furthermore, L_n is naturally contained in $\widehat{L}_n$.) At this point, we state the Baker-Campbell-Hausdorff formula (or briefly, BCH) (see [10, page 178], [15, Chapter 5, Theorem 5.19], [3, Chapter 8]). It is

$$X \circ Y \;=\; X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,Y,Y] - \tfrac{1}{12}[X,Y,X] + \cdots ,$$

where each term on the right-hand side is a rational multiple of a Lie commutator $[Z_1, \ldots, Z_m]$, $m \in \mathbb{N}$, and each Z_i is X or Y and only finitely many terms of each length occur. Note that the right-hand side of the aforementioned formula is an infinite sum. The formula states that $X \circ Y$ belongs to the completion of the free Lie algebra freely generated by the set $\{X, Y\}$ with respect to its lower central series.

The BCH formula defines an associative operation in $\widehat{L}_n$. For $x, y \in \widehat{L}_n$ the operation $x \circ y$ is defined by

$$x \circ y = x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,y,x] - \frac{1}{12}[x,y,y] + \cdots$$

(see, for example, [15, Chapter 5, page 369]). (We remark that if L is a Lie algebra and the right-hand side of the BCH formula has a meaning for all x, y in L, then L becomes a group with respect to $\circ$.) It is easily verified that, for $x, y \in \widehat{L}_n$,

$$(x, y) = [x, y] + \frac{1}{2}[x, y, x] + \frac{1}{2}[x, y, y] + \cdots .$$

Let L be a relatively free Lie algebra of rank n, with $n \geq 2$, freely generated by the set $\{t_1, \ldots, t_n\}$, and let $\widehat{L}$ be the completion of L with respect to the lower central series. Write π_n^* for the natural epimorphism from $\widehat{L}_n$ onto $\widehat{L}$. Then π_n^* preserves multiplication $\circ$ and hence is a group homomorphism from $(\widehat{L}_n, \circ)$ into $(\widehat{L}, \circ)$. Thus, for $i, j \in \{1, \ldots, n\}$, we have

$$\pi_n^*(\ell_i \circ \ell_j) = t_i \circ t_j = t_i + t_j + \frac{1}{2}[t_i, t_j] + \frac{1}{12}[t_i, t_j, t_i] - \frac{1}{12}[t_i, t_j, t_j] + \cdots .$$

Notice that if L is nilpotent, then $\widehat{L} = L$.

In section 2, for a torsion-free finitely generated nilpotent group G, we naturally associate four finitely generated nilpotent Lie algebras, namely, $\mathcal{L}_K(G)$, $\mathrm{grad}^{(\ell)}(\mathcal{L}_K(G))$, $\mathrm{grad}^{(g)}(\exp \mathcal{L}_K(G))$ and $L_K(G)$. One of our main aims, in this paper, is to prove the following theorem.

Theorem A *(I). Let $\mathfrak{T}_c$ be a torsion-free variety of nilpotent groups of class at most c. For a positive integer n, with $n \geq 2$, let $F_n(\mathfrak{T}_c)$ be the relatively free group of rank n in $\mathfrak{T}_c$. Then the Lie algebra $\mathcal{L}_K(F_n(\mathfrak{T}_c))$ is relatively free in some variety of nilpotent Lie algebras, and $\mathcal{L}_K(F_n(\mathfrak{T}_c)) \cong L_K(F_n(\mathfrak{T}_c))$ as Lie algebras in a natural way. Moreover, $F_n(\mathfrak{T}_c)$ is a Magnus nilpotent group.*

(II). Let L be a relatively free nilpotent Lie algebra over $\mathbb{Q}$ of finite rank with a free generating set $\mathcal{X}$. Give on L the structure of a group R by means of the Baker-Campbell-Hausdorff formula. Let H be the subgroup of R generated by the set $\mathcal{X}$. Then H is relatively free in some variety of nilpotent groups; freely generated by the set $\mathcal{X}$, H is Magnus and $L \cong \mathcal{L}_{\mathbb{Q}}(H) \cong L_{\mathbb{Q}}(H)$ as Lie algebras in a natural way.

In [12], Kofinas and Papistas develop a method, by making use of Theorem A (II), in order to study the automorphism group of a relatively free nilpotent Lie algebra over $\mathbb{Q}$ of finite rank.

Corollary 1.1 *Let G be a torsion-free finitely generated nilpotent group and K a field of characteristic zero. Then*

(I). $\mathrm{grad}^{(\ell)}(\mathcal{L}_K(G)) \cong \mathrm{grad}^{(g)}(\exp \mathcal{L}_K(G)) \cong L_K(G)$ *as Lie algebras.*

(II). If G is relatively free, then

$$\mathcal{L}_K(G) \cong \mathrm{grad}^{(\ell)}(\mathcal{L}_K(G))$$

as Lie algebras.

Now let (X_1, d_1) and (X_2, d_2) be metric spaces. A map $f : X_1 \to X_2$ is called (λ, ε)-*quasi-isometry* if there exist constants $\lambda \geq 1$, $\varepsilon \geq 0$ and $C \geq 0$ such that

(I).
$$\frac{1}{\lambda} d_1(x, y) - \varepsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \varepsilon$$
for all $x, y \in X_1$;

(II). every point of X_2 lies in the C-neighbourhood of the image of f.

Notice that the above map f need not be continuous. Every finitely generated group G with generating set S can be turned into a metric space with the *word metric* in G. If $F(S)$ is the free group with generating set S and $\phi : F(S) \to G$ is the natural projection, then the word metric in G is the metric obtained by defining $d_S(g_1, g_2)$ to be the shortest word in the pre-image of $g_1^{-1} g_2$ under ϕ. For more on quasi-isometries the reader could consult [5]. The metric space (G, d_S) does not depend on the choice of S. In fact if S' is a different generating set for G then (G, d_S) and $(G, d_{S'})$ are quasi-isometric.

Corollary 1.2 *Let G_1 and G_2 be torsion-free finitely generated nilpotent groups which are quasi-isometric. If G_1 and G_2 are relatively free of finite rank, then they are isomorphic.*

Obviously, Corollary 1.2 implies that the simply connected nilpotent Lie groups L_1 and L_2 given by the Mal'cev completion of G_1 and G_2 respectively, are isomorphic. It is a well know conjecture whether the same result is true if we drop the relative freeness assumption (see [14, 7]).

Throughout this paper, we write $\mathfrak{L}$ for a residually torsion-free nilpotent variety of groups (that is, a variety with its free groups of arbitrary rank to be residually torsion-free

nilpotent groups). Recall that a group G is called *residually torsion-free nilpotent* if for any $g \in G \setminus \{1\}$ there exists a normal subgroup N_g such that $g \notin N_g$ and G/N_g is a torsion-free nilpotent group. For a positive integer n, with $n \geq 2$, we write $G_n = F_n(\mathfrak{L})$. In section 3.3, for G_n, we naturally associate two finitely generated Lie algebras, $\mathcal{L}_K(G_n)$ and $L_K(G_n)$. The following result extends Theorem A to residually torsion-free nilpotent groups.

Theorem B *(I). Let $\mathfrak{L}$ be a residually torsion-free nilpotent variety of groups. For a positive integer n, with $n \geq 2$, let G_n be a relatively free group of rank n in $\mathfrak{L}$. Then the Lie algebra $\mathcal{L}_K(G_n)$ is relatively free, and $\mathcal{L}_K(G_n) \cong L_K(G_n)$ as Lie algebras in a natural way.*

(II). Let L be a relatively free Lie algebra over $\mathbb{Q}$ of rank n, with $n \geq 2$, freely generated by the set $\mathcal{X}$. Let $\widehat{L}$ be the completion of L with respect to the lower central series. Give on $\widehat{L}$ the structure of a group $\widehat{R}$ via the Baker-Campbell-Hausdorff formula, and consider L as a Lie subalgebra of $\widehat{L}$. Let H be the subgroup of $\widehat{R}$ generated by the set $\mathcal{X}$. Then H is a relatively free residually torsion-free nilpotent group of rank n freely generated by the set $\mathcal{X}$. A Lie algebra $\Lambda_{\mathbb{Q}}(H)$ over $\mathbb{Q}$, associated with H, is constructed such that $L \cong \Lambda_{\mathbb{Q}}(H)$ as Lie algebras in a natural way. Furthermore, $\Lambda_{\mathbb{Q}}(H)$ is a homomorphic image of $\mathcal{L}_{\mathbb{Q}}(H)$.

Notice that there is some overlap of Theorem B with some results in [3]. Namely, the Lie algebra $L_K(G_n)$ is proved to be relatively free [3, Theorem 10, page 278], and H to be a relatively free residually torsion-free nilpotent group of rank n [3, Theorem 8, pages 276-277. See, also, Comments on pages 296-297].

The paper is organized as follows: In section 2, for any torsion-free finitely generated nilpotent group G, four finitely generated nilpotent Lie algebras are naturally defined, and for a finitely generated nilpotent Lie algebra, we naturally associate a torsion-free finitely generated nilpotent group by means of the Baker-Campbell-Hausdorff formula. Some auxiliary lemmas are proved in section 3. Moreover, relatively free groups and relatively Lie algebras are studied. In section 4, we prove Theorems A and B, and Corollaries 1.1 and 1.2. An example of a finitely generated Magnus nilpotent group G, not relatively free, such that $\mathcal{L}_{\mathbb{Q}}(G) \ncong L_{\mathbb{Q}}(G)$ is given in section 5.

2 Nilpotent groups and Lie algebras

Let H be a nilpotent group and denote by $\tau(H)$ the set of all elements of finite order in H. Then $\tau(H)$ is a subgroup of H, it is characteristic in H, and $H/\tau(H)$ is torsion-free. For a group N and a positive integer i, let π_i be the natural mapping from N onto $N/\gamma_i(N)$. Since $N/\gamma_i(N)$ is nilpotent, $\tau(N/\gamma_i(N))$ is a group. Write $\tau_i(N)$ for the complete inverse image of $\tau(N/\gamma_i(N))$ in N via π_i i.e. $\tau_i(N) = \{g \in N : g^n \in \gamma_i(N)$ for some integer $n\}$. We call $\tau_i(N)$ the *isolator* of $\gamma_i(N)$ in N. Note that $\tau_i(N)/\gamma_i(N) = \tau(N/\gamma_i(N))$ for all i.

Let $\mathfrak{L}$ be a residually torsion-free nilpotent variety of groups. For a positive integer n, with $n \geq 2$, we write $G_n = F_n(\mathfrak{L})$. The condition of being residually torsion-free nilpotent is equivalent to $\cap_{i\geq 1}\tau_i(G_n) = \{1\}$. Since each $\tau_i(G_n)$ is a fully invariant subgroup of G_n, it is easily verified that there are no repetitions of terms of the series $\{\tau_i(G_n)\}_{i\geq 1}$. Notice that $(\tau_i(G_n), \tau_j(G_n)) \leq \tau_{i+j}(G)$ for all i, j. For each positive integer i, we write $L_i(G_n)$ for the quotient group $\tau_i(G_n)/\tau_{i+1}(G_n)$. Form the (restricted) direct sum of abelian groups $L(G_n) = \oplus_{i\geq 1}L_i(G_n)$ and give on it a structure of a Lie ring by defining a Lie multiplication $[a\tau_{i+1}(G_n), b\tau_{j+1}(G_n)] = (a, b)\tau_{i+j+1}(G_n)$, where $a\tau_{i+1}(G_n)$ and $b\tau_{j+1}(G_n)$ are the images of the elements $a \in \tau_i(G_n)$ and $b \in \tau_j(G_n)$ in the quotient groups $L_i(G_n)$ and $L_j(G_n)$, respectively, and $(a, b)\tau_{i+j+1}(G_n)$ is the image of the group commutator (a, b) in the quotient group $L_{i+j}(G_n)$. Multiplication is then extended to $L(G_n)$ by linearity. Form the tensor product of K with $L(G_n)$ over $\mathbb{Z}$ and write $L_K(G_n) = K \otimes_{\mathbb{Z}} L(G_n)$. Then $L_K(G_n)$ has the structure of a Lie algebra with $\lambda(\lambda' \otimes a) = \lambda\lambda' \otimes a$ and $[\lambda \otimes a, \lambda' \otimes a'] = \lambda\lambda' \otimes [a, a']$ for all $\lambda, \lambda' \in K$ and $a, a' \in L(G_n)$. Since each $L_i(G_n)$ is a free $\mathbb{Z}$-module with a basis, say X_i, every element of $K \otimes_{\mathbb{Z}} L_i(G_n)$ may be written uniquely as a K-linear combination of elements $1 \otimes x$ with $x \in X_i$. We write $L_{i,K}(G_n)$ for the vector space over K spanned by any $\mathbb{Z}$-basis of $L_i(G_n)$. Thus we may regard $L_i(G_n)$ as a subset of $L_{i,K}(G_n)$ and so, we regard $L(G_n)$ as a subset of $L_K(G_n)$. Furthermore $L_K(G_n) = \oplus_{i\geq 1}L_{i,K}(G_n)$.

For the rest of this section, G denotes a torsion-free finitely generated nilpotent group of class c. The series

$$G = \tau_1(G) \supset \tau_2(G) \supset \cdots \supset \tau_c(G) \supset \tau_{c+1}(G) = \{1\} \qquad (1)$$

is a characteristic central series of G with $\tau_i(G)/\tau_{i+1}(G)$ torsion-free for all i, $1 \leq i \leq c$ (see [18, page 49]). Form the direct sum of abelian groups $L(G) = \oplus_{t=1}^{c}\tau_t(G)/\tau_{t+1}(G)$ and

let $L_K(G) = \oplus_{t=1}^{c}(K \otimes \tau_t(G)/\tau_{t+1}(G))$. As before, we give on $L_K(G)$ the structure of a Lie algebra. Let n_i denote the rank of the free abelian group $L_i(G)$. For $i = 1, \dots, c$, let $f(i) = n_1 + \cdots + n_{i-1}$, with $n_0 = 0$ and $n_1 = n$. Let $X_i = \{a_{f(i)+1}, \dots, a_{f(i+1)}\}$ be a subset of $\tau_i(G)$ such that the set $\{a_{f(i)+1}\tau_{i+1}(G), \dots, a_{f(i+1)}\tau_{i+1}(G)\}$ is a $\mathbb{Z}$-basis of $L_i(G)$. We refine the series (1) of G to obtain a central series

$$G = \mathcal{G}_1 \supset \cdots \supset \mathcal{G}_{f(i)+1} \supset \cdots \supset \mathcal{G}_{f(i+1)} \supset \cdots \supset \mathcal{G}_{f(c+1)} \supset \mathcal{G}_{f(c+1)+1} = \{1\}, \qquad (2)$$

with $i = 1, \dots, c$, such that $a_{f(i)+j}$ is a representative in G of a generating element of $\mathcal{G}_{f(i)+j}$ modulo $\mathcal{G}_{f(i)+j+1}$, with $j = 1, \dots, n_i$. (The length $f(c+1)$ is an invariant for G. It is called the *Hirsch number* of G; denoted by $\mathcal{H}(G)$.) Following [11], we call the aforementioned central series of G, a *$\mathcal{G}$-series* of G. (In [11] it is called $\mathcal{F}$-series.) Thus every element g of G may be written uniquely in the form

$$g = a_1^{\beta_1} \cdots a_{n_1}^{\beta_{n_1}} \cdots a_{f(i)+1}^{\beta_{f(i)+1}} \cdots a_{f(i+1)}^{\beta_{f(i+1)}} \cdots a_{f(c)+1}^{\beta_{f(c)+1}} \cdots a_{f(c+1)}^{\beta_{f(c+1)}}, \qquad (3)$$

where $\beta_j \in \mathbb{Z}$. In what follows we assume that the aforementioned series (2) and the elements $a_1, \dots, a_{f(c+1)}$ in (3) have been selected. The set $\{a_1, \dots, a_{f(c+1)}\}$ is called a *canonical basis* (or, *Mal'cev basis*) of G.

Let $\Gamma = KG$ be the group algebra of G over K, and let Δ be the augmentation ideal of Γ. It has been proved in [11, Theorem 4.3] that $\cap_{i\geq 1}\Delta^i = \{0\}$. Take the set $\{\Delta^i\}_{i\geq 1}$ as a fundamental system of neighbourhoods of the element 0 in Γ; then a sequence $b_1, b_2, \dots, b_n, \dots$ of elements of Γ converges to $b \in \Gamma$ if, for every i, there exists an integer $n(i)$ such that $n > n(i)$ implies that $b_n - b \in \Delta^i$. So $\{\Delta^i\}$ induce a topology on Γ and let Γ^* be the completion of Γ in this topology, and Δ^* be the completion of Δ. We may consider Γ^* to be the algebra of all *formal power series* a^* of the form $a^* = \alpha_0 + \sum \alpha_k d_k$, where $\alpha_0, \alpha_k \in K$, $k = 1, 2, \dots$, and $d_k \in \Delta^k$, while Δ^* consists of all elements a^* with $\alpha_0 = 0$. We identify Γ with its isomorphic image in Γ^*. Define $\exp : \Delta^* \longrightarrow 1 + \Delta^*$ and $\log : 1 + \Delta^* \longrightarrow \Delta^*$ as in [11].

We associate with Δ^* the Lie algebra $\Lambda^* = (\Delta^*)_L$ in the usual way by defining the binary operation of commutation in Λ^* by means of $[x^*, y^*] = x^*y^* - y^*x^*$ for all $x^*, y^* \in \Delta^*$. For a positive integer c, with $c \geq 3$, we define the left normed Lie product $[y_1^*, \dots, y_{c-1}^*, y_c^*] = [[y_1^*, \dots, y_{c-1}^*], y_c^*]$. It is proved in [11] that $\cap_{i\geq 1}(\Delta^*)^i = \{0\}$. Furthermore it is easily verified that if Θ^* is an ideal of Δ^*, and $M^* = (\Theta^*)_L$, then M^* is an ideal of the Lie algebra Λ^*. Thus

we have $\Lambda^* = (\Delta^*)_L \supset (\Delta^*)_L^2 \supset \cdots$. For a positive integer i, let $\gamma_i(\Lambda^*)$ denote the i-th term of the lower central series of Λ^*. Then, for all i, $\gamma_i(\Lambda^*) \subseteq (\Delta^*)_L^i$. Therefore $\cap_{i\geq 1}\gamma_i(\Lambda^*) = \{0\}$ and we obtain $\Lambda^* = \gamma_1(\Lambda^*) \supset \gamma_2(\Lambda^*) \supset \cdots \supset \gamma_i(\Lambda^*) \supset \cdots$. If $v_k \in \gamma_k(\Lambda^*)$ for $k = 1, 2, \ldots$, then the infinite series $v_1 + v_2 + \cdots + v_k + \cdots$ is an element of Λ^* (see [11, Lemma 6.1]).

The BCH formula reveals the intimate connection between the group $1 + \Delta^*$ and the Lie algebra Λ^*. For $X = 1 + x^*$ and $Y = 1 + y^* \in 1 + \Delta^*$,

$$\log XY = \log X + \log Y + \frac{1}{2}[\log X, \log Y] + \frac{1}{12}[\log X, \log Y, \log Y] - \cdots$$

where each term on the right-hand side is a rational multiple of a Lie commutator $[Z_1, \ldots, Z_m]$, $m \in \mathbb{N}$, and each Z_i is $\log X$ or $\log Y$ and only finitely many terms of each length occur. Note that the right-hand side of the aforementioned formula is an infinite sum and therefore convergent. Since log and exp are mutually inverse bijections, we define an operation on $\Lambda^* = (\Delta^*)_L$ as follows: Let $u, v \in \Lambda^*$. Then there exists unique X and Y in $1 + \Delta^*$ such that $\log X = u$ and $\log Y = v$. Define

$$\begin{aligned} u \circ_G v &= \log X \circ_G \log Y \\ &= \log XY \\ &= u + v + \tfrac{1}{2}[u, v] + \tfrac{1}{12}[u, v, v] - \cdots . \end{aligned}$$

Notice that $(\Lambda^*, \circ_G)$ is a group. We write $\circ$ instead of $\circ_G$ if it is clear in the context. Let $\mathcal{L}_K(G)$ be the vector space over K spanned by all $\log g$ with $g \in G$. Then $\mathcal{L}_K(G)$ is a nilpotent Lie subalgebra of class c of Λ^*, and the set $\{\log a_1, \ldots, \log a_{f(c+1)}\}$ is a K-basis of $\mathcal{L}_K(G)$ (see [11, Theorem 7.3]). Thus $\dim \mathcal{L}_K(G) = \mathcal{H}(G)$. Notice that $\mathcal{L}_K(G) = K \otimes_{\mathbb{Q}} \mathcal{L}_{\mathbb{Q}}(G)$. Observe that $\exp \mathcal{L}_K(G)$ is a subgroup of $1 + \Delta^*$ and $(\exp u)(\exp v) = \exp(u \circ v)$ for all $u, v \in \mathcal{L}_K(G)$. Furthermore $(\mathcal{L}_K(G), \circ)$ is a subgroup of $(\Lambda^*, \circ)$. It is easily verified that $\mathcal{L}_K(G)$ is isomorphic as group to $\exp \mathcal{L}_K(G)$ by a mapping sending u to $\exp u$ for all $u \in \mathcal{L}_K(G)$. Form the direct sum of the vector spaces over K

$$\mathrm{grad}^{(\ell)}(\mathcal{L}_K(G)) = \oplus_{t=1}^{c} \gamma_t(\mathcal{L}_K(G))/\gamma_{t+1}(\mathcal{L}_K(G)).$$

The Lie multiplication in $\mathcal{L}_K(G)$ induces a Lie algebra structure on $\mathrm{grad}^{(\ell)}(\mathcal{L}_K(G))$. Namely, for all i, j,

$$[u + \gamma_{i+1}(\mathcal{L}_K(G)), v + \gamma_{j+1}(\mathcal{L}_K(G))] = [u, v] + \gamma_{i+j+1}(\mathcal{L}_K(G)),$$

where $u \in \gamma_i(\mathcal{L}_K(G))$, $v \in \gamma_j(\mathcal{L}_K(G))$ and $[u, v] \in \gamma_{i+j}(\mathcal{L}_K(G))$. Multiplication is then extended to $\mathrm{grad}^{(\ell)}(\mathcal{L}_K(G))$ by linearity.

Let G be a torsion-free finitely generated nilpotent group. A *Mal'cev completion* of G is a torsion-free radicable nilpotent group R containing G and such that for all $a \in R$, there exists $r \in \mathbb{N}$ such that $a^r \in G$. (A group G is said to be *radicable* if for every x in G and an arbitrary natural number m, there exists y in G such that $y^m = x$. If G is a torsion-free nilpotent group, $x, y \in G$, and $x^m = y^m$ for some $m \in \mathbb{N}$, then $x = y$ (see [13, page 247].) If G is a subgroup of a torsion-free radicable nilpotent group S, then S contains a Mal'cev completion of G. Any two Mal'cev completions of G are isomorphic by an isomorphism which fixes G pointwise (that is, R is unique up to isomorphism). We note that $\exp \mathcal{L}_{\mathbb{Q}}(G)$ is a Mal'cev completion of G (see [4, 18]).

Let L be a finitely generated nilpotent Lie algebra over $\mathbb{Q}$ and let L' denote the derived algebra of L. Let $n = \dim L/L'$, and let $h_1, \ldots, h_n$ be elements of L such that the set $\{h_1 + L', \ldots, h_n + L'\}$ is a $\mathbb{Q}$-basis of L/L'. We assert that L is generated by the set $\{h_1, \ldots, h_n\}$ as Lie algebra. Indeed, let $\{y_1, \ldots, y_m\}$ be a generating set of L and m be the smallest number of generators. It is easily shown that $m = n$. Thus $\{y_1 + L', \ldots, y_n + L'\}$ is a $\mathbb{Q}$-basis of L/L'. Let A be the Lie subalgebra of L generated by the set $\{h_1, \ldots, h_n\}$. To show that $A = L$ it is enough to prove that $L \subseteq A$. For $j = 1, \ldots, n$,

$$y_j = \sum_{i=1}^{n} \alpha_{ij} h_i + v_j,$$

where $\alpha_{ij} \in \mathbb{Q}$, $v_j \in L'$. Since L is nilpotent, we can finally express each Lie commutator $[y_{i_1}, \ldots, y_{i_\kappa}]$, $\kappa \in \mathbb{N}$, as a $\mathbb{Q}$-linear combination of Lie commutators $[h_{j_1}, \ldots, h_{j_\ell}]$, $\ell \in \mathbb{Q}$. Therefore $L \subseteq A$ and so, $L = A$. Hence L is generated by the set $\{h_1, \ldots, h_n\}$. We give on L, by means of BCH formula, the structure of a group, denoted by R. It is well known that R is a torsion-free radicable nilpotent group. Let H be the subgroup of R generated by the set $\{h_1, \ldots, h_n\}$. Then R is a Mal'cev completion of H, and $H' = \tau_2(H)$ (see [6, Proof of Theorem B, page 457]). Since both $\exp \mathcal{L}_{\mathbb{Q}}(H)$ and R are Mal'cev completions of H, we obtain $R \cong \exp \mathcal{L}_{\mathbb{Q}}(H)$ as groups (see, for example, [18, Chapter 6, Corollary 4]) and so, by the Mal'cev correspondence, $L \cong \mathcal{L}_{\mathbb{Q}}(H)$ as Lie algebras by an isomorphism sending h_i to $\log h_i$, with $i = 1, \ldots, n$. We state the above observations as lemma.

Lemma 2.1 *Let L be a finitely generated nilpotent Lie algebra over $\mathbb{Q}$, and let $h_1, \ldots, h_n$ be elements of L such that the set $\{h_1 + L', \ldots, h_n + L'\}$ is a $\mathbb{Q}$-basis of L/L'. Then L is generated by the set $\{h_1, \ldots, h_n\}$. Consider L as a group by means of the Baker-Campbell-Hausdorff*

formula, denoted by R. Let H be the subgroup of R generated by the set $\{h_1, \ldots, h_n\}$. Then R is a Mal'cev completion of H, H is torsion-free finitely generated nilpotent group of class c, $H' = \tau_2(H)$, and $L \cong \mathcal{L}_{\mathbb{Q}}(H)$ as Lie algebras. $\square$

3 Groups and Lie algebras

3.1 Some auxiliary Lemmas

Throughout this section we shall give some auxiliary results helping us to prove our main results. Let L be a Lie algebra. For a positive integer i, let $\gamma_i(L)$ be the i-th term of the lower central series of L. Write $L' = \gamma_2(L)$. The following result is well-known (and it is easily proved).

Lemma 3.1 *Let G be a torsion-free finitely generated nilpotent group. Then $\gamma_i(G)\tau_{i+1}(G)/\tau_{i+1}(G)$ has finite index in $\tau_i(G)/\tau_{i+1}(G)$.* $\square$

For a proof of the following result, we refer the reader to [9, Chapter VIII, Lemma 9.4, page 330].

Lemma 3.2 *Suppose that G is a group and that L is a Lie ring. Suppose that for $i \geq 1$, σ_i is a homomorphism of $\gamma_i(G)/\gamma_{i+1}(G)$ onto an additive subgroup L_i of L such that $L = L_1 + L_2 + \cdots$. Suppose further that if $x \in \gamma_i(G)$, $y \in \gamma_j(G)$,*

$$[\sigma_i(x\gamma_{i+1}(G)), \sigma_j(y\gamma_{j+1}(G))] = \sigma_{i+j}((x,y)\gamma_{i+j+1}(G)),$$

where $(x,y) = x^{-1}y^{-1}xy$.

i) If G is generated by a set X, L is the Lie ring generated by the set

$$Y = \{\sigma_1(x\gamma_2(G)) : x \in X\}.$$

ii) For $r = 1, 2, \ldots$, $\gamma_r(L) = L_r + L_{r+1} + \cdots$, and $\gamma_r(L)/\gamma_{r+1}(L)$ is isomorphic to $L_r/(L_r \cap \gamma_{r+1}(L))$. $\square$

Let G be a torsion-free finitely generated nilpotent group of class c. Let $a_1, \ldots, a_n$ be elements of G such that $\{a_1\tau_2(G), \ldots, a_n\tau_2(G)\}$ is a $\mathbb{Z}$-basis of $G/\tau_2(G)$. Since $(G/G')/(\tau_2(G)/G')$ is naturally isomorphic to $G/\tau_2(G)$, and $G/\tau_2(G)$ is a free $\mathbb{Z}$-module

of rank n, there are elements $v_1, \ldots, v_n \in \tau_2(G)$ such that the set $\{a_1 v_1 G', \ldots, a_n v_n G'\}$ is a part of a generating set of G/G'. Let $y_{n+1}, \ldots, y_m$ be elements of $\tau_2(G)$ subject to $\{a_1 v_1 G', \ldots, a_n v_n G', y_{n+1} G', \ldots, y_m G'\}$ is a generating set of G/G'. Set $Y = \{a_1 v_1, \ldots, a_n v_n, y_{n+1}, \ldots, y_m\}$. Since G/G' is finitely generated, we obtain $\gamma_i(G)/\gamma_{i+1}(G)$ is finitely generated for all i. In fact, the set of all group commutators of the form $(u_1, \ldots, u_i)$, with $u_1, \ldots, u_i \in Y$, modulo $\gamma_{i+1}(G)$, generates $\gamma_i(G)/\gamma_{i+1}(G)$. It is easily verified that $(z_1, \ldots, z_{i-1}) \in \tau_i(G)$, with $i \geq 2$, if some $z_j \in \{y_{n+1}, \ldots, y_m\}$ with $j = 1, \ldots, i-1$. Thus the set $Z_i = \{(z_1, \ldots, z_i)\tau_{i+1}(G) : z_1, \ldots, z_i \in \{a_1, \ldots, a_n\}\}$ generates $\gamma_i(G)\tau_{i+1}(G)/\tau_{i+1}(G)$. Write $L^{(S)}(G) = \oplus_{1 \leq i \leq c} L_i^{(S)}(G)$, with $L_i^{(S)}(G) = \gamma_i(G)\tau_{i+1}(G)/\tau_{i+1}(G)$. It is easily checked that $L^{(S)}(G)$ is a Lie subring of $L(G)$. For each positive integer i, let π_i be the natural mapping from $\gamma_i(G)/\gamma_{i+1}(G)$ onto $L_i^{(S)}(G)$. Since, for $x \in \gamma_i(G)$ and $y \in \gamma_j(G)$,

$$[\pi_i(x\gamma_{i+1}(G)), \pi_j(y\gamma_{j+1}(G))] = \pi_{i+j}((x, y)\gamma_{i+j+1}(G)),$$

we obtain from Lemma 3.2 that $L^{(S)}(G)$ is the Lie ring generated by the set $\{a_1\tau_2(G), \ldots, a_n\tau_2(G)\}$. Since $L_i^{(S)}(G)$ is free abelian, we obtain $K \otimes L_i^{(S)}(G)$ is a finite-dimensional vector space over K. Since the set of all elements of the form $1 \otimes z$, with $z \in Z_i$, spans $K \otimes L_i^{(S)}(G)$, we get there exists a subset X_i of Z_i such that the set of all elements of the form $1 \otimes x$, with $x \in X_i$, is a K-basis of $K \otimes L_i^{(S)}(G)$. Since $L_i^{(S)}(G)$ has finite index in $\tau_i(G)/\tau_{i+1}(G)$ (Lemma 3.1), and $\tau_i(G)/\tau_{i+1}(G)$ is free abelian, we obtain $K \otimes L_i^{(S)}(G) = L_{i,K}(G)$. Hence it is easy to verify the following result.

Lemma 3.3 *Let G be a torsion-free finitely generated nilpotent group, and let $a_1, \ldots, a_n$ be elements of G such that $\{a_1\tau_2(G), \ldots, a_n\tau_2(G)\}$ is a $\mathbb{Z}$-basis of $G/\tau_2(G)$. Then $\{a_1\tau_2(G), \ldots, a_n\tau_2(G)\}$ is a generating set of $L_K(G)$, and $\{(a_1\tau_2(G)) + L_K(G)', \ldots, (a_n\tau_2(G)) + L_K(G)'\}$ is a K-basis for $L_K(G)/L_K(G)'$.* $\square$

The following result gives a generating set of $\mathcal{L}_K(G)$ as Lie algebra with respect to a given $\mathbb{Z}$-basis of $G/\tau_2(G)$.

Lemma 3.4 *Let G be a torsion-free finitely generated nilpotent group, and let $a_1, \ldots, a_n$ be elements of G such that $\{a_1\tau_2(G), \ldots, a_n\tau_2(G)\}$ is a $\mathbb{Z}$-basis of $G/\tau_2(G)$. Then $\{\log a_1, \ldots, \log a_n\}$ generates $\mathcal{L}_K(G)$ as Lie algebra.*

Proof. Let $a_1, \ldots, a_n$ be elements of G such that $\{a_1\tau_2(G), \ldots, a_n\tau_2(G)\}$ is a $\mathbb{Z}$-basis of $G/\tau_2(G)$. Choose a canonical basis $\{a_1, \ldots, a_n, \ldots, a_{f(c+1)}\}$ of G where c is the nilpotency class of G. Let $\mathcal{A}$ be the Lie subalgebra of $\mathcal{L}_K(G)$ generated by the set $\{\log a_1, \ldots, \log a_n\}$. We claim that $\mathcal{A} = \mathcal{L}_K(G)$. It is enough to show that $\mathcal{L}_K(G) \subseteq \mathcal{A}$. In fact it is enough to show that $\log g \in \mathcal{A}$ for all $g \in G$. Since $\mathcal{A}$ is a Lie algebra and $\{\log a_1, \ldots, \log a_{f(c+1)}\}$, is a K-basis of $\mathcal{L}_K(G)$ (as vector space), it is enough to show that $\log g \in \mathcal{A}$ for all $g \in \{a_1, \ldots, a_{f(c+1)}\}$. Let $y_{n+1}, \ldots y_m$ be elements of $\tau_2(G)$ subject to $\{a_1v_1G', \ldots, a_nv_nG', y_{n+1}G', \ldots, y_mG'\}$, with $v_1, \ldots, v_n \in \tau_2(G)$, is a generating set for G/G'. The set $\{(z_1, \ldots, z_i)\tau_{i+1}(G) : z_1, \ldots, z_i \in \{a_1, \ldots, a_n\}\}$ generates $\gamma_i(G)\tau_{i+1}(G)/\tau_{i+1}(G)$. Since $\gamma_i(G)\tau_{i+1}(G)/\tau_{i+1}(G)$ has finite index in $\tau_i(G)/\tau_{i+1}(G)$ (by Lemma 3.1), we obtain $a_{f(i)+j}^{m_{ij}} = g_{ij}u_{i+1,j}$ for some $m_{ij} \in \mathbb{Z}$, g_{ij} is a product of group commutators of the form $(z_1, \ldots, z_i)$ and $z_1, \ldots, z_i \in \{a_1, \ldots, a_n\}$, and $u_{i+1,j} \in \tau_{i+1}(G)$ $j = 1, \ldots, n_i$. For $i = c$, we have

$$a_{f(c)+j}^{m_{ij}} = g_{cj} \tag{4}$$

for $j = 1, \ldots, n_c$. By the BCH formula, we have $\log g \in \mathcal{A}$ for all $g \in \gamma_c(G)$. Applying the BCH formula on (4) and since $\log g_{cj} \in \mathcal{A}$, we obtain $\log a_{f(c)+1}, \ldots, \log a_{f(c+1)} \in \mathcal{A}$, and hence $\log g \in \mathcal{A}$ for all $g \in \tau_c(G)$. Suppose that for all κ, with $i < \kappa \leq c$, $\log a_{f(\kappa)+1}, \ldots, \log a_{f(c+1)} \in \mathcal{A}$. Let $h \in \tau_k(G)$. Since G is nilpotent and $\cap_{j\geq 1}\tau_j(G) = \{1\}$, we write $h = g_\kappa g_{\kappa+1} \cdots g_c$ with $g_j \in \tau_j(G) \setminus \tau_{j+1}(G)$ and $j = \kappa, \ldots, c$. Note that

$$h = a_{f(\kappa)+1}^{\beta_{f(\kappa)+1}} \cdots a_{f(\kappa+1)}^{\beta_{f(\kappa+1)}} \cdots a_{f(c)+1}^{\beta_{f(c)+1}} \cdots a_{f(c+1)}^{\beta_{f(c+1)}}. \tag{5}$$

Applying the BCH formula on (5), our hypothesis and since $\mathcal{A}$ is a Lie algebra, we have $\log h \in \mathcal{A}$ for all $h \in \tau_\kappa(G)$. In particular, for $\kappa = i + 1$,

$$\log h \in \mathcal{A} \ \text{ for all } \ h \in \tau_{i+1}(G). \tag{6}$$

For $z_1, \ldots, z_i \in \{a_1, \ldots, a_n\}$, the BCH formula gives

$$\log(z_1, \ldots, z_i) = [\log z_1, \ldots, \log z_i] + \sum_\mu r_\mu d_\mu \tag{7}$$

where each d_μ is a repeated Lie commutator of length at least $i + 1$ in the arguments $\log z_1, \ldots, \log z_i$ each of which appears at least once; and the coefficients $r_\mu \in \mathbb{Q}$ (see [18, Corollary 2, page 102]. The arguments given in the proof of Corollary 2 can be adopted here.)

Once more, applying the BCH formula on a product of group commutators of length i, using the equation (7) and the fact that $\mathcal{A}$ is a Lie algebra generated by the set $\{\log a_1, \ldots, \log a_n\}$, we obtain

$$\log(\prod_{\text{finite}} (z_1, \ldots, z_i)^{\nu}) \in \mathcal{A}, \tag{8}$$

where $\nu \in \mathbb{Z}$. But $a_{f(i)+j}^{m_{ij}} = g_{ij} u_{(i+1)j}$ for some $m_{ij} \in \mathbb{Z}$, g_{ij} is a product of group commutators of the form $(z_1, \ldots, z_i)$, with $z_1, \ldots, z_i \in \{a_1, \ldots, a_n\}$, and $u_{(i+1)j} \in \tau_{i+1}(G)$ $j = 1, \ldots, n_i$. Apply the BCH formula on $a_{f(i)+j}^{m_{ij}} = g_{ij} u_{(i+1)j}$. By the equations (7) and (8) and since $\mathcal{A}$ is a Lie algebra, we obtain $\log a_{f(i)+1}, \ldots, \log a_{f(i+1)} \in \mathcal{A}$. Thus $\mathcal{L}_K(G) \subseteq \mathcal{A}$ and so $\mathcal{L}_K(G) = \mathcal{A}$. $\square$

From the proof of Lemma 3.4, we obtain the following result.

Corollary 3.1 *Let G be a torsion-free finitely generated nilpotent group of class c, and let $a_1, \ldots, a_n$ be elements of G such that $\{a_1\tau_2(G), \ldots, a_n\tau_2(G)\}$ is a $\mathbb{Z}$-basis of $G/\tau_2(G)$. Let $\{a_1, a_2, \ldots, a_{f(c+1)}\}$ be a canonical basis of G. Then for $j \geq f(t) + 1$, with $t \geq 2$, $\log a_j \in \gamma_t(\mathcal{L}_K(G))$.* $\square$

Lemma 3.5 *Let G be a torsion-free finitely generated nilpotent group of class c. For a positive integer t, with $2 \leq t \leq c$, let $\tau_t(G)$ be the isolator of $\gamma_t(G)$ in G. Let π_t be the natural mapping from G onto $G/\tau_t(G)$. Then there exists a Lie algebra epimorphism ξ_{π_t} from $\mathcal{L}_K(G)$ onto $\mathcal{L}_K(G/\tau_t(G))$ such that, for all $g \in G$, $\xi_{\pi_t}(\log g) = \log \pi_t(g)$. Furthermore $\ker\xi_{\pi_t} = \mathcal{L}_K(\tau_t(G))$.*

Proof. For $H \in \{G, G/\tau_t(G)\}$, we write $\Gamma_H = KH$ for the group algebra of H over K, and Δ_H for the augmentation ideal of Γ_H. Notice that H is a torsion-free finitely generated nilpotent group. Let Γ_H^* be the algebra of all formal power series a^* of the form $a^* = \alpha_0 + \sum \alpha_k d_k$, where $\alpha_0, \alpha_k \in K$, $k = 1, 2, \ldots$, and $d_k \in \Delta_H^k$. In addition, we write Δ_H^* for the subalgebra of Γ_H^* consisting of all elements a^* with $\alpha_0 = 0$. We associate with Δ_H^* the Lie algebra $\Lambda_H^* = (\Delta_H^*)_L$. Let $h \in H$. The natural mapping π_t induces an algebra epimorphism $\pi_{t,K}^*$ from Δ_G^* onto $\Delta_{G/\tau_t(G)}^*$ such that $\pi_{t,K}^*(\log g) = \log \pi_t(g)$ for all $g \in G$. It is easily checked that $\pi_{t,K}^*$ induces a Lie algebra epimorphism from Λ_G^* onto $\Lambda_{G/\tau_t(G)}^*$. Hence we obtain a Lie algebra epimorphism ξ_{π_t} from $\mathcal{L}_K(G)$ onto $\mathcal{L}_K(G/\tau_t(G))$ such that $\xi_{\pi_t}(\log g) = \log \pi_t(g)$ for

all $g \in G$. But

$$\begin{aligned}\dim \ker \xi_{\pi_t} &= \dim \mathcal{L}_K(G) - \dim \mathcal{L}_K(G/\tau_t(G)) \\ &= \mathcal{H}(G) - \mathcal{H}(G/\tau_t(G)) \\ &= \mathcal{H}(\tau_t(G)) \\ &= \dim \mathcal{L}_K(\tau_t(G)).\end{aligned}$$

Let $v \in \mathcal{L}_K(\tau_t(G))$. Then $v = k_1 \log v_1 + \cdots + k_s \log v_s$ with $v_1, \ldots, v_s \in \tau_t(G)$ and $k_1, \ldots, k_s \in K$. Since $\xi_{\pi_t}(v) = k_1 \log \pi_t(v_1) + \cdots + k_s \log \pi_t(v_s) = 0$, we obtain $v \in \ker \xi_{\pi_t}$. Therefore $\mathcal{L}_K(\tau_t(G)) \subseteq \ker \xi_{\pi_t}$ and so, $\ker \xi_{\pi_t} = \mathcal{L}_K(\tau_t(G))$. $\square$

The following result gives us information about the terms of the lower central series of $\mathcal{L}_K(G)$.

Lemma 3.6 *Let G be a torsion-free finitely generated nilpotent group of class c. Then $\gamma_t(\mathcal{L}_K(G)) = \mathcal{L}_K(\tau_t(G))$ for all t, with $t = 1, \ldots, c$.*

Proof. Since $\mathcal{L}_K(\tau_t(G))$ has a K-basis the set $\{\log a_{f(t)+1}, \ldots, \log a_{f(c+1)}\}$, and by Corollary 3.1, it is enough to show that $\gamma_t(\mathcal{L}_K(G)) \subseteq \mathcal{L}_K(\tau_t(G))$. By Lemma 3.5, $\ker \xi_{\pi_t} = \mathcal{L}_K(\tau_t(G))$ and so, it is enough to show that $\gamma_t(\mathcal{L}_K(G)) \subseteq \ker \xi_{\pi_t}$. Let $v \in \gamma_t(\mathcal{L}_K(G))$. By Lemma 3.4, v is written as a K-linear combination of Lie commutators of the form $[\log a_{i_1}, \ldots, \log a_{i_\kappa}]$ with $t \leq \kappa \leq c$. The BCH formula gives

$$[\log a_{i_1}, \ldots, \log a_{i_\kappa}] = \log(a_{i_1}, \ldots, a_{i_\kappa}) + \sum_i s_i \log v_i, \tag{9}$$

where each v_i is a left normed group commutator of length at least $\kappa + 1$ in the arguments $a_{i_1}, \ldots, a_{i_\kappa}$, each of which appears at least once; and the coefficients belong to $\mathbb{Q}$. Applying ξ_{π_t} to the equation (9), we get $[\log a_{i_1}, \ldots, \log a_{i_\kappa}] \in \ker \xi_{\pi_t} = \mathcal{L}_K(\tau_t(G))$. Therefore $\gamma_t(\mathcal{L}_K(G)) \subseteq \ker \xi_{\pi_t}$ and so, $\gamma_t(\mathcal{L}_K(G)) = \mathcal{L}_K(\tau_t(G))$. $\square$

The following result is probably well-known.

Lemma 3.7 *For any torsion-free finitely generated nilpotent group G of class c,* $\mathrm{grad}^{(\ell)}(\mathcal{L}_K(G))$ *is isomorphic to $L_K(G)$ as Lie algebras under the mapping ϕ sending* $\log a_{f(i)+j} + \gamma_{i+1}(\mathcal{L}_K(G))$ *to $a_{f(i)+j}\tau_{i+1}(G)$ for all $i = 1, \ldots, c$, $j = 1, \ldots, n_i$.*

Proof. By the proof of Lemma 3.6, the set $\{\log a_{f(t)+1}, \ldots, \log a_{f(c+1)}\}$ is a K-basis of $\gamma_t(\mathcal{L}_K(G)) = \mathcal{L}_K(\tau_t(G))$. Notice that

$$\gamma_t(\mathcal{L}_K(G)) = \mathrm{span}\{\log a_{f(t)+1}, \ldots, \log a_{f(t+1)}\} \oplus \gamma_{t+1}(\mathcal{L}_K(G)).$$

Since

$$\gamma_t(\mathcal{L}_K(G))/\gamma_{t+1}(\mathcal{L}_K(G)) \cong K \otimes \tau_t(G)/\tau_{t+1}(G)$$

as vector spaces for $t = 1, \ldots, c$, we obtain the mapping ϕ from $\mathrm{grad}^{(\ell)}(\mathcal{L}_K(G))$ to $L_K(G)$ sending $\log a_{f(i)+j} + \gamma_{i+1}(\mathcal{L}_K(G))$ to $a_{f(i)+j}\tau_{i+1}(G)$ for all $i = 1, \ldots, c$, $j = 1, \ldots, n_i$, is a K-linear isomorphism. By the BCH formula, we obtain

$$\phi([\log a_{f(i)+\mu} + \gamma_{i+1}(\mathcal{L}_K(G)), \log a_{f(j)+\nu} + \gamma_{j+1}(\mathcal{L}_K(G))]) =$$
$$\phi([\log a_{f(i)+\mu}, \log a_{f(j)+\nu}] + \gamma_{i+j+1}(\mathcal{L}_K(G))) = \phi(\log(a_{f(i)+\mu}, a_{f(j)+\nu}) + \gamma_{i+j+1}(\mathcal{L}_K(G)).$$

Write

$$(a_{f(i)+\mu}, a_{f(j)+\nu}) = a_{f(i+j)+1}^{m_1} \cdots a_{f(i+j+1)}^{m_{n_{i+j}}} v,$$

where $m_1, \ldots, m_{n_{i+j}} \in \mathbb{Z}$, $v \in \tau_{i+j+1}(G)$. Then, by BCH formula,

$$\phi([\log a_{f(i)+\mu} + \gamma_{i+1}(\mathcal{L}_K(G)), \log a_{f(j)+\nu} + \gamma_{j+1}(\mathcal{L}_K(G))]) =$$
$$\phi(m_1 \log a_{f(i+j)+1} + \cdots + m_{n_{i+j}} \log a_{f(i+j+1)} + \gamma_{i+j+1}(\mathcal{L}_K(G))) =$$
$$m_1 a_{f(i+j)+1}\tau_{i+j+1}(G) + \cdots + m_{n_{i+j}} a_{f(i+j+1)}\tau_{i+j+1}(G) = a_{f(i+j)+1}^{m_1} \cdots a_{f(i+j+1)}^{m_{n_{i+j}}}\tau_{i+j+1}(G)$$
$$= (a_{f(i)+\mu}, a_{f(j)+\nu})\tau_{i+j+1}(G) = [\phi(\log a_{f(i)+\mu} + \gamma_{i+1}(\mathcal{L}_K(G))), \phi(\log a_{f(j)+\nu} + \gamma_{j+1}(\mathcal{L}_K(G)))])$$

for all $i, j \in \{1, \ldots, c\}$, $\mu \in \{1, \ldots, n_i\}$ and $\nu \in \{1, \ldots, n_j\}$. Thus ϕ is a Lie algebra isomorphism. □

3.2 Relatively free groups

Let $\mathfrak{N}_c$ be the variety of nilpotent groups of class at most c, and let $\mathfrak{T}_c$ be a torsion-free subvariety of class at most c of $\mathfrak{N}_c$. For the rest of this section, for positive integers n and c, with $n \geq 2$, we write $F_{n,c} = F_n(\mathfrak{N}_c)$ and $G = F_n(\mathfrak{T}_c)$. The groups $F_{n,c}$ and G are freely generated by the set $\{x_1, \ldots, x_n\}$, $x_i = f_i\mathfrak{N}_c(F_n)$, $i = 1, \ldots, n$, and the set $\{y_1, \ldots, y_n\}$, $y_i = f_i\mathfrak{T}_c(F_n)$, $i = 1, \ldots, n$, respectively. Since $\mathfrak{T}_c$ is a subvariety of $\mathfrak{N}_c$, the natural map π from $F_{n,c}$ to G is surjective. Since $\ker\pi = \mathfrak{T}_c(F_n)/\mathfrak{N}_c(F_n)$, we obtain $\ker\pi$ is a fully invariant subgroup of $F_{n,c}$. Hence $\ker\pi$ is a verbal subgroup of $F_{n,c}$. Let μ and ν be the Hirsch numbers of $F_{n,c}$ and G, respectively. Since $F_{n,c}$ is finitely generated nilpotent, $\ker\pi$ is a (torsion-free) finitely generated nilpotent group of class at most c. Since $F_{n,c}/\ker\pi \cong G$ and G is torsion-free, it follows from a result of Hirsch (see [8, Theorem 2.312], also, [11, Theorem 2.3]) that $\ker\pi$ has Hirsch number $\mu - \nu$. Since $L_K(F_{n,c})$ is a free nilpotent Lie algebra of class c freely generated

by the set $\{x_1F'_{n,c}, \ldots, x_nF'_{n,c}\}$ (see, for example, [19]), $\mathcal{L}_K(F_{n,c})$ is a nilpotent Lie algebra of class c and $\{\log x_1, \ldots, \log x_n\}$ generates $\mathcal{L}_K(F_{n,c})$ (by Lemma 3.4), we obtain the mapping χ from the set $\{x_1F'_{n,c}, \ldots, x_nF'_{n,c}\}$ into $\mathcal{L}_K(F_{n,c})$ sending $x_iF'_{n,c}$ to $\log x_i$, $i = 1, \ldots, n$, extends uniquely to a Lie algebra epimorphism. Since $\dim L_K(F_{n,c}) = \dim \mathcal{L}_K(F_{n,c}) = \mu$, we obtain χ is one-to-one and so, χ is a Lie algebra isomorphism. We summarize the above observations as follows.

Lemma 3.8 *For positive integers n and c, with $n \geq 2$, let $F_{n,c}$ be the free nilpotent group of rank n and class c; freely generated by the set $\{x_1, \ldots, x_n\}$. Then $\mathcal{L}_K(F_{n,c})$ is a free nilpotent Lie algebra of rank n and class c; freely generated by the set $\{\log x_1, \ldots, \log x_n\}$.* $\square$

The next result gives us a way of how a group homomorphism of $F_{n,c}$ onto G can define a Lie algebra homomorphism from $\mathcal{L}_K(F_{n,c})$ onto $\mathcal{L}_K(G)$. The proof of the following result is similar to the proof given in Lemma 3.5.

Lemma 3.9 *Let M be a torsion-free finitely generated nilpotent group of class c such that $M/\tau_2(M)$ is a free abelian group of rank n, with $n \geq 2$. Let φ be any group homomorphism from $F_{n,c}$ into M, and let ψ_φ be the mapping from the set $\{\log x_1, \ldots, \log x_n\}$ into $\mathcal{L}_K(M)$ defined by $\psi_\varphi(\log x_i) = \log \varphi(x_i)$ for $i = 1, \ldots, n$. Then ψ_φ extends uniquely to a Lie algebra homomorphism from $\mathcal{L}_K(F_{n,c})$ into $\mathcal{L}_K(M)$ and, for all $u \in F_{n,c}$, $\psi_\varphi(\log u) = \log \varphi(u)$.* $\square$

Lemma 3.10 *For positive integers n and c, with $n \geq 2$, let $G = F_n(\mathfrak{T}_c)$, with $n \geq 2$, and let π be the natural mapping from $F_{n,c}$ onto G. Then $\ker\pi \subseteq F'_{n,c}$, and $\mathcal{L}_K(\ker\pi) \subseteq \mathcal{L}_K(F'_{n,c}) = \mathcal{L}_K(F_{n,c})'$.*

Proof. First we shall show that $\ker\pi \subseteq F'_{n,c}$. To get a contradiction, we assume that $\ker\pi \nsubseteq F'_{n,c}$. Thus $(F_{n,c}/\ker\pi)' = F'_{n,c}\ker\pi/\ker\pi$. Since G/G' is free abelian of rank n, and $G/G' \cong F_{n,c}/F'_{n,c}\ker\pi$, we get $F_{n,c}/F'_{n,c}\ker\pi$ is free abelian of rank n. Since $F_{n,c}/F'_{n,c}$ is free abelian of rank n, we obtain $F'_{n,c}\ker\pi/F'_{n,c}$ is not a trivial free abelian subgroup of $F_{n,c}/F'_{n,c}$ which is a contradiction. Therefore $\ker\pi \subseteq F'_{n,c}$. Thus $\mathcal{L}(\ker\pi) \subseteq \mathcal{L}_K(F'_{n,c})$. By Lemma 3.6, we obtain $\mathcal{L}_K(F'_{n,c}) = \mathcal{L}_K(F_{n,c})'$. Hence $\mathcal{L}_K(\ker\pi) \subseteq \mathcal{L}_K(F_{n,c})'$. $\square$

For a torsion-free finitely generated nilpotent group G of class c, we write $\mathcal{L}_t(G)$ for the vector subspace of $\mathcal{L}_K(G)$ spanned by all Lie commutators of the form $[\log a_{i_1}, \ldots, \log a_{i_t}]$ with $i_1, \ldots, i_t \in \{1, \ldots, n\}$.

Lemma 3.11 *For positive integers n and c, with $n \geq 2$, let $G = F_n(\mathfrak{T}_c)$ and let π be the natural mapping from $F_{n,c}$ onto G. Then*

$$\mathcal{L}_K(\ker\pi) = \oplus_{t=2}^{c}(\mathcal{L}_K(\ker\pi) \cap \mathcal{L}_t(F_{n,c})).$$

Proof. By Lemma 3.10, and since $\mathcal{L}_K(F_{n,c}) = \oplus_{t=1}^{c}\mathcal{L}_K(F_{n,c})$, we have

$$\mathcal{L}_K(\ker\pi) \subseteq \mathcal{L}_K(F_{n,c})' = \oplus_{t=2}^{c}\mathcal{L}_t(F_{n,c}).$$

Let $w \in \mathcal{L}_K(\ker\pi)$. For a fixed i, with $i = 1, \dots, n$, express w as a sum

$$w = w_0 + w_1 + \cdots + w_s$$

where the number of times that $\log x_i$ occurs in w_j is j, $j = 0, \dots, s$. Pick distinct non-zero integer numbers $\alpha_0, \alpha_1, \dots, \alpha_s$. Then

$$w(\log x_1, \dots, \alpha_j \log x_i, \dots, \log x_n) = \sum_{\kappa=0}^{s} \alpha_j^{\kappa} w_\kappa(\log x_1, \dots, \log x_n).$$

Now the determinant

$$\begin{vmatrix} 1 & \alpha_0 & \alpha_0^2 & \cdots & \alpha_0^s \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^s \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_s & \alpha_s^2 & \cdots & \alpha_s^s \end{vmatrix},$$

is a Vandermonte determinant with value $\prod_{i<j}(\alpha_i - \alpha_j)$ so is non-zero. Consequently, each w_j is a $\mathbb{Q}$-linear combination of the elements $w(\log x_1, \dots, \alpha_j \log x_i, \dots, \log x_n)$. For the fixed i and $j = 0, \dots, s$, let $\varphi_{i,j}$ be the mapping from the set $\{x_1, \dots, x_n\}$ into $F_{n,c}$ defined by $\varphi_{i,j}(x_\kappa) = x_i^{\alpha_j}$ if $i = \kappa$ and $\varphi_{i,j}(x_\kappa) = x_\kappa$ if $i \neq \kappa$. Since $F_{n,c}$ is relatively free on the set $\{x_1, \dots, x_n\}$, $\varphi_{i,j}$ extends uniquely to a group endomorphism of $F_{n,c}$. By Lemma 3.9 (for $M = F_{n,c}$), $\psi_{\varphi_{i,j}}$ is a Lie algebra endomorphism of $\mathcal{L}_K(F_{n,c})$. Write

$$w = \lambda_1 \log u_1 + \cdots + \lambda_s \log u_s,$$

where $\lambda_1, \dots, \lambda_s \in K$ and $u_1, \dots, u_s \in \ker\pi$. Applying $\psi_{\varphi_{i,j}}$ on w, and since $\ker\pi$ is a fully invariant subgroup of $F_{n,c}$, we obtain $\psi_{\varphi_{i,j}}(w) \in \mathcal{L}_K(\ker\pi)$. Hence

$$w(\log x_1, \dots, \alpha_j \log x_i, \dots, \log x_n) \in \mathcal{L}_K(\ker\pi)$$

for $j = 0, \dots, s$. Therefore, for $\kappa = 0, \dots, s$, we have

$$w_\kappa(\log x_1, \dots, \log x_n) \in \mathcal{L}_K(\ker\pi).$$

Consequently, each w_i is a $\mathbb{Q}$-linear combination of the elements

$$w(\log x_1, \dots, \alpha_j \log x_i, \dots, \log x_n),$$

for $j = 0, \dots, s$. If we repeat the process on each w_i using different $\log x_\kappa$ (with $\kappa \neq i$), then eventually we obtain each homogeneous component of w belongs to $\mathcal{L}_K(\ker\pi)$. Therefore if $w \in \mathcal{L}_K(\ker\pi)$ then the homogeneous components of w belong to $\mathcal{L}_K(\ker\pi)$ and so, we obtain the required result. $\Box$

Proposition 3.1 *For all positive integers n and c, with $n \geq 2$, $\mathcal{L}_K(F_{n,c}/\ker\pi)$ is isomorphic to $\mathcal{L}_K(F_{n,c})/\mathcal{L}_K(\ker\pi)$ as a Lie algebra. Furthermore, $\mathcal{L}_{\mathbb{Q}}(\ker\pi)$ is a fully invariant ideal of $\mathcal{L}_{\mathbb{Q}}(F_{n,c})$.*

Proof. By Lemma 3.9 (for $M = G$), the mapping ψ_π from the set $\{\log x_1, \dots, \log x_n\}$ into $\mathcal{L}_K(G)$ defined by $\psi_\pi(\log x_i) = \log \pi(x_i)$ for $i = 1, \dots, n$ extends uniquely to a Lie algebra homomorphism from $\mathcal{L}_K(F_{n,c})$ into $\mathcal{L}_K(G)$ and, for all $u \in F_{n,c}$, $\psi_\pi(\log u) = \log \pi(u)$. Since π is onto and by the definition of $\mathcal{L}_K(G)$, we obtain ψ_π is onto. First we shall show that $\ker\psi_\pi = \mathcal{L}_K(\ker\pi)$. Let $w \in \mathcal{L}_K(\ker\pi)$. Then

$$w = \kappa_1 \log v_1 + \cdots + \kappa_s \log v_s,$$

where $v_1, \dots, v_s \in \ker\pi$ and $\kappa_1, \dots, \kappa_s \in K$. Applying ψ_π, we have

$$\psi_\pi(w) = \kappa_1 \log \pi(v_1) + \cdots + \kappa_s \log \pi(v_s) = 0.$$

Therefore $w \in \ker\psi_\pi$. Hence

$$\mathcal{L}_K(\ker\pi) \subseteq \ker\psi_\pi. \tag{10}$$

But $\dim \mathcal{L}_K(\ker\pi) = \mathcal{H}(\ker\pi) = \mu - \nu$. In addition, $\dim(\ker\psi_\pi) = \dim \mathcal{L}_K(F_{n,c}) - \dim \mathcal{L}_K(G) = \mathcal{H}(F_{n,c}) - \mathcal{H}(G) = \mu - \nu$. By the equation (10), we obtain $\ker\psi_\pi = \mathcal{L}_K(\ker\pi)$.

Next we show that $\mathcal{L}_{\mathbb{Q}}(\ker\pi)$ is a fully invariant ideal of $\mathcal{L}_{\mathbb{Q}}(F_{n,c})$. Let ξ be a Lie algebra homomorphism of $\mathcal{L}_{\mathbb{Q}}(F_{n,c})$, and let $\xi(\log x_i) = u_i$ for $i = 1, \dots, n$. Since $\exp \mathcal{L}_{\mathbb{Q}}(F_{n,c})$

is a Mal'cev completion of $F_{n,c}$ there are positive integers m_i such that $\exp(m_i u_i) \in F_{n,c}$ for $i = 1, \ldots, n$. Let φ be the endomorphism of $F_{n,c}$ satisfying the conditions $\varphi(x_i) = \exp(m u_i)$, with m the least common multiple of $m_1, \ldots, m_n$ for $i = 1, \ldots, n$. (Notice that $\exp(m u_i) = 1$ if and only if $u_i = 0$.) Since $\ker\pi$ is a fully invariant subgroup of $F_{n,c}$, we have $\varphi(v) \in \ker\pi$ for all $v \in \ker\pi$. Let $w \in \mathcal{L}_{\mathbb{Q}}(\ker\pi)$ and, by Lemma 3.11, we assume that w is a homogeneous element. Write

$$w = w(\log x_1, \ldots, \log x_n) = \lambda_1 \log v_1 + \cdots + \lambda_s \log v_s,$$

where $\lambda_1, \ldots, \lambda_s \in \mathbb{Q}$ and $v_1, \ldots, v_s \in \ker\pi$. By Lemma 3.9 (for $M = F_{n,c}$), $\psi_\varphi(\log u) = \log \varphi(u)$ for all $u \in F_{n,c}$. Applying ψ_φ on w, we obtain $\psi_\varphi(w) \in \mathcal{L}_{\mathbb{Q}}(\ker\pi)$. Since ψ_φ is a Lie algebra homomorphism of $\mathcal{L}_{\mathbb{Q}}(F_{n,c})$, we obtain $\lambda w(u_1, \ldots, u_n) \in \mathcal{L}_{\mathbb{Q}}(\ker\pi)$ for some non-zero integer λ. Hence $\xi(w) \in \mathcal{L}_{\mathbb{Q}}(\ker\pi)$ for all homogeneous elements $w \in \mathcal{L}_{\mathbb{Q}}(\ker\pi)$. Since for any element w in $\mathcal{L}_{\mathbb{Q}}(\ker\pi)$ the homogeneous components of w belong to $\mathcal{L}_{\mathbb{Q}}(\ker\pi)$ (by Lemma 3.11) and ξ is a Lie algebra homomorphism, we obtain $\mathcal{L}_{\mathbb{Q}}(\ker\pi)$ is a fully invariant ideal of $\mathcal{L}_{\mathbb{Q}}(F_{n,c})$. $\square$

Proposition 3.2 *For any relatively free of finite rank n torsion-free nilpotent group G, its Lie algebra $\mathcal{L}_K(G)$ is relatively free in some variety of nilpotent Lie algebras.*

Proof. Since $\mathcal{L}_{\mathbb{Q}}(F_{n,c})$ is a free nilpotent Lie algebra (by Lemma 3.8) and any fully invariant ideal of $\mathcal{L}_{\mathbb{Q}}(F_{n,c})$ is verbal (The arguments given in the proof of Theorem 13.31 in [16] are still valid for relatively free Lie algebras.), it follows from Proposition 3.1 that $\mathcal{L}_{\mathbb{Q}}(F_{n,c}/\ker\pi)$ is relatively free. Since $G \cong F_{n,c}/\ker\pi$, we obtain $\mathcal{L}_{\mathbb{Q}}(G)$ is relatively free. Recall that $\mathcal{L}_K(G) = K \otimes_{\mathbb{Q}} \mathcal{L}_{\mathbb{Q}}(G)$. Since relatively freeness is preserved by a field extension, we obtain the required result (see [1, Chapter 14]). $\square$

Theorem 3 *For any relatively free, torsion-free nilpotent group G of finite rank n, $\mathcal{L}_K(G)$ is isomorphic to $L_K(G)$ as a Lie algebra via an isomorphism η sending $\log y_i$ to $y_i G'$, $i = 1, \ldots, n$.*

Proof. Write $G = F_n(\mathfrak{T}_c)$ for some torsion-free subvariety $\mathfrak{T}_c$ of class at most c of $\mathfrak{N}_c$. It is freely generated by the set $\{y_1, \ldots, y_n\}$, where $y_i = f_i \mathfrak{T}_c(F_n)$. Then the factor group G/G' is either a free abelian group freely generated by the set $\{y_1 G', \ldots, y_n G'\}$ or it is the abelian group of exponent $m > 0$ every element of whose basis $\{y_1 G', \ldots, y_n G'\}$ has order m (see

[16, page 11]). If G/G' has finite exponent m, then, since G is nilpotent of class c, we obtain G has finite exponent dividing m^c (see, for example, [18, page 13]) which is a contradiction. Therefore G/G' is a free abelian group of rank n. Hence $\tau_2(G) = G'$.

By Lemma 3.3, $L_K(G)$ is generated as Lie algebra by the set $\{y_1G', \ldots, y_nG'\}$. In addition, $L_K(G)$ is a nilpotent Lie algebra of class c. Then $L_K(G) \in \mathfrak{M}_c$. Since $\mathcal{L}_K(F_{n,c})$ is free in $\mathfrak{M}_c$ freely generated by the set $\{\log x_1, \ldots, \log x_n\}$ (by Lemma 3.8), the mapping ρ from $\{\log x_1, \ldots, \log x_n\}$ into $L_K(G)$ defined by $\rho(\log x_i) = y_iG'$, $i = 1, \ldots, n$, extends uniquely to a Lie algebra epimorphism from $\mathcal{L}_K(F_{n,c})$ onto $L_K(G)$. We claim that $\rho(\mathcal{L}_K(\ker\pi)) = \{0\}$. Recall that $\mathcal{L}_K(\ker\pi) = \ker\psi_\pi$ (see the proof of Proposition 3.1), where ψ_π is the Lie algebra epimorphism from $\mathcal{L}_K(F_{n,c})$ onto $\mathcal{L}_K(G)$ such that $\psi_\pi(\log x_i) = \log \pi(x_i) = \log y_i$, $i = 1, \ldots, n$. Thus, by Lemma 3.11,

$$\ker\psi_\pi = \oplus_{t=2}^{c}(\ker\psi_\pi \cap \mathcal{L}_t(F_{n,c})). \tag{11}$$

To prove that $\rho(\ker\psi_\pi) = \{0\}$ it is enough to show that $\rho(v) = 0$ for all $v \in \ker\psi_\pi$. By the equation (11), it is enough to prove that $\rho(v) = 0$ for all homogeneous elements in $\ker\psi_\pi$. For any homogeneous Lie commutator w in $\mathcal{L}_K(F_{n,c})$, we write $\widetilde{w}$ for the Lie commutator in $L_K(G)$ obtained from w by replacing the Lie multiplication in $\mathcal{L}_K(F_{n,c})$ by the Lie multiplication in $L_K(G)$, and, in addition, by replacing each $\log x_i$ by y_iG' with $i = 1, \ldots, n$. Let $v \in \ker\psi_\pi \cap \mathcal{L}_t(F_{n,c})$ for some $t \geq 2$. Write

$$v = \sum \beta_{(i_1,\ldots,i_t)}[\log x_{i_1}, \ldots, \log x_{i_t}],$$

where $\beta_{(i_1,\ldots,i_t)} \in K$. Then

$$0 = \psi_\pi(v) = \sum \beta_{(i_1,\ldots,i_t)}[\log y_{i_1}, \ldots, \log y_{i_t}]. \tag{12}$$

Since

$$[\log y_{i_1}, \ldots, \log y_{i_t}] = \log(y_{i_1}, \ldots, y_{i_t}) + u,$$

where $u \in \mathcal{L}_K(\tau_{t+1}(G))$, we obtain from the equation (12)

$$\sum \beta_{(i_1,\ldots,i_t)} \log(y_{i_1}, \ldots, y_{i_t}) \in \mathcal{L}(\tau_{t+1}(G)). \tag{13}$$

Choose a canonical basis $\{a_1, \ldots, a_{f(c+1)}\}$ of G with $a_i = y_i$, $i = 1, \ldots, n$. Write $(y_{i_1}, \ldots, y_{i_t}) = a_{f(t)+1}^{m_{f(t)+1}} \cdots a_{f(t+1)}^{m_{f(t+1)}} x$, where $x \in \tau_{t+1}(G)$. By the BCH formula and since $\mathcal{L}_K(\tau_{t+1}(G))$ has a

K-basis the set $\{\log a_{f(t+1)+1}, \ldots, \log a_{f(c+1)}\}$, we get from (13)

$$\sum \beta_{(i_1,\ldots,i_t)}(\sum_{k=f(t)+1}^{f(t+1)} m_k \log a_k) = 0.$$

Let ζ be the K-linear isomorphism from $\mathcal{L}_K(G)$ onto $L_K(G)$ such that $\zeta(\log a_{f(i)+j}) = a_{f(i)+j}\tau_{i+1}(G)$ for all $i = 1, \ldots, c$ and $j = 1, \ldots, n_i$. Since $\mathcal{L}_K(\tau_{t+1}(G))$ has a K-basis the set $\{\log a_{f(t+1)+1}, \ldots, \log a_{f(c+1)}\}$ and since $\gamma_{t+1}(L_K(G)) = \oplus_{j=t+1}^{c} L_{j,K}(G)$, we have $\zeta(\mathcal{L}_K(\tau_{t+1}(G))) = \gamma_{t+1}(L_K(G))$. Therefore

$$\sum \beta_{(i_1,\ldots,i_t)}(\prod_{k=f(t)+1}^{f(t+1)} (a_k\tau_{t+1}(G))^{m_k}) = 0. \tag{14}$$

Since

$$[y_{i_1}G', \ldots, y_{i_t}G'] = (y_{i_1}, \ldots, y_{i_t})\tau_{t+1}(G) = \prod_{k=f(t)+1}^{f(t+1)} (a_k\tau_{t+1}(G))^{m_k},$$

we obtain from (14) $\sum \beta_{(i_1,\ldots,i_t)}[y_{i_1}G', \ldots, y_{i_t}G'] = 0$. Thus

$$\rho(v) = \widetilde{v} = \sum \beta_{(i_1,\ldots,i_t)}[y_{i_1}G', \ldots, y_{i_t}G'] = 0.$$

Therefore $\rho(\mathcal{L}_K(\ker\pi)) = \{0\}$. Hence $\mathcal{L}_K(\ker\pi) \subseteq \ker\rho$. Since $\dim\ker\rho = \dim\mathcal{L}_K(F_{n,c}) - \dim L_K(G) = \mu - \nu = \dim\mathcal{L}_K(\ker\pi)$, we obtain $\mathcal{L}_K(\ker\pi) = \ker\rho$. Therefore $\mathcal{L}_K(F_{n,c})/\mathcal{L}_K(\ker\pi) \cong L_K(G)$ by a Lie algebra isomorphism ρ_1 such that $\rho_1(\log x_i + \mathcal{L}(\ker\pi)) = y_iG'$, $i = 1, \ldots, n$. By the proof Proposition 3.1, $\mathcal{L}_K(F_{n,c})/\mathcal{L}_K(\ker\pi) \cong \mathcal{L}_K(G)$ by a Lie algebra isomorphism $\psi_{\pi,1}$ such that $\psi_{\pi,1}(\log x_i + \mathcal{L}_K(\ker\pi)) = \log y_i$, $i = 1, \ldots, n$. Let $\eta = \rho_1 \circ \psi_{\pi,1}^{-1}$. Then η is a Lie algebra isomorphism from $\mathcal{L}_K(G)$ into $L_K(G)$ such that $\eta(\log y_i) = y_iG'$ for $i = 1, \ldots, n$. $\Box$

Let L be a relatively free nilpotent Lie algebra over $\mathbb{Q}$ of finite rank n; freely generated by the set $\{h_1, \ldots, h_n\}$. It is easily verified that the set $\{h_1 + L', \ldots, h_n + L'\}$ is a $\mathbb{Q}$-basis of L/L'. Give on L, by means of the Baker-Campbell-Hausdorff formula, the structure of a group denoted by R. Let H be the subgroup of R generated by the set $\{h_1, \ldots, h_n\}$ and let c be the nilpotency class of L. Notice that the nilpotency class of H is c as well. Since H is generated by n elements and it is nilpotent of class c (that is, $H \in \mathfrak{N}_c$), and since $F_{n,c}$ is relatively free in $\mathfrak{N}_c$, the map τ from $F_{n,c}$ into H sending x_i to h_i, $i = 1, \ldots, n$, is a group epimorphism. By Lemma 2.1, $H' = \tau_2(H)$. Thus, the first part of Lemma 3.10 shows

that $\ker\tau \subseteq F'_{n,c}$. By Lemma 3.9 (for $M = H$), τ induces a Lie algebra homomorphism ψ_τ from $\mathcal{L}_{\mathbb{Q}}(F_{n,c})$ into $\mathcal{L}_{\mathbb{Q}}(H)$ such that $\psi_\tau(\log u) = \log\tau(u)$ for all $u \in F_{n,c}$. Note that the set $\{h_1H', \ldots, h_nH'\}$ is a $\mathbb{Z}$-basis for H/H' (see [6, Proof of Theorem B, page 457]). By Lemma 3.9, ψ_τ is surjective. Thus $\mathcal{L}_{\mathbb{Q}}(F_{n,c})/\ker\psi_\tau \cong \mathcal{L}_{\mathbb{Q}}(H)$. By applying similar arguments as in the proof of Proposition 3.1, we obtain $\mathcal{L}_{\mathbb{Q}}(\ker\tau) = \ker\psi_\tau$ and so, $\mathcal{L}_{\mathbb{Q}}(F_{n,c})/\mathcal{L}_{\mathbb{Q}}(\ker\tau) \cong \mathcal{L}_{\mathbb{Q}}(H)$. By Lemma 2.1, $L \cong \mathcal{L}_{\mathbb{Q}}(H)$ and so, $\mathcal{L}_{\mathbb{Q}}(H)$ is relatively free. Thus $\mathcal{L}_{\mathbb{Q}}(F_{n,c})/\mathcal{L}_{\mathbb{Q}}(\ker\tau)$ is relatively free. It is easy to verify that $\mathcal{L}_{\mathbb{Q}}(\ker\tau)$ is a fully invariant ideal of $\mathcal{L}_{\mathbb{Q}}(F_{n,c})$, and $\ker\tau$ is a fully invariant subgroup of $F_{n,c}$. So, we obtain the following result.

Proposition 3.3 *Let τ and H be as above.*
(i) $\mathcal{L}_{\mathbb{Q}}(F_{n,c})/\mathcal{L}_{\mathbb{Q}}(\ker\tau) \cong \mathcal{L}_{\mathbb{Q}}(H)$.
(ii) $\mathcal{L}_{\mathbb{Q}}(\ker\tau)$ is a fully invariant ideal of $\mathcal{L}_{\mathbb{Q}}(F_{n,c})$.
(iii) $\ker\tau$ *is a fully invariant subgroup of* $F_{n,c}$. $\square$

3.3 Relatively free Lie algebras

An inverse system (G_i, φ_{ij}) of sets indexed by a directed nonempty set I consists of a family $\{G_i|\ i \in I\}$ of sets and a family $\{\varphi_{ij} : G_j \to G_i|\ i, j \in I, i \leq j\}$ of maps such that φ_{ii} is the identity map Id_{G_i} for each i and $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ whenever $i \leq j \leq k$. We shall call a family $\{\psi_i : Y \to G_i|\ i \in I\}$ of maps *compatible* to the inverse system (G_i, ϕ_{ij}) if $\varphi_{ij}\psi_j = \psi_i$ whenever $i \leq j$. An *inverse limit* $(\widetilde{G}, \varphi_i)$ of an inverse system (G_i, φ_{ij}) of sets is a set $\widetilde{G}$ together with a compatible family $\{\varphi_i : \widetilde{G} \to G_i\}$ of maps with the following universal property: whenever $(\psi_i : Y \to G_i)$ is a compatible family of maps from a set Y, there is a unique map $\psi : Y \to \widetilde{G}$ such that $\varphi_i\psi = \psi_i$ for each i. Write $C = \prod_{i\in I} G_i$ for the cartesian product of all sets G_i, and for each i write π_i for the projection map from C to G_i. Define

$$\widetilde{G} = \{c \in C : \varphi_{ij}\pi_j(c) = \pi_i(c) \ \text{ for all } \ i, j \text{ with } i \leq j\}$$

and $\varphi_i = \pi_i|_{\widetilde{G}}$ for each i. Then $(\widetilde{G}, \varphi_i)$ is an inverse limit of (G_i, φ_{ij}), denoted by $\varprojlim G_i$. Suppose each G_i is a Lie algebra and each φ_{ij} is a Lie algebra homomorphism. Then C is a Lie algebra, and it is easy to see that $\widetilde{G}$ is a Lie subalgebra of C. The coordinate projections π_i are obviously homomorphisms in this case.

Let $x_i = f_i\mathfrak{L}(F_n)$, with $i = 1, \ldots, n$. That is, the set $\{x_1, \ldots, x_n\}$ is a basis (i.e. a free generating set) for G_n. Since G_n is relatively free and $\tau_{c+1}(G_n)$ is a fully invariant

subgroup of G_n, we obtain $\tau_{c+1}(G_n)$ is verbal. Write $G_n = F_n/\mathfrak{L}(F_n)$ and let N_{c+1} be the complete inverse image in F_n of $\tau_{c+1}(G_n)$. Then $N_{c+1}/\mathfrak{L}(F_n) = \tau_{c+1}(G_n)$. It is easily verified that N_{c+1} is a fully invariant subgroup of F_n and so, N_{c+1} is verbal. That is, there exists a set Ω_{c+1} of words such that $N_{c+1} = \Omega_{c+1}(F_n)$. But then $N_{c+1}/\mathfrak{L}(F_n) = \Omega_{c+1}(F_n/\mathfrak{L}(F_n))$ (see [16], the proof of 13.31 Theorem, page 10]). Thus $G_n/\tau_{c+1}(G_n) \cong F_n/N_{c+1}$ and so, $G_n/\tau_{c+1}(G_n)$ is a relatively free torsion-free nilpotent group of rank n and nilpotency class at most c. We write $G_{n,c} = G_n/\tau_{c+1}(G_n)$ and claim that $G_{n,c}$ has class exactly c. Indeed, to get a contradiction we assume that $\gamma_d(G_n) \subseteq \tau_{c+1}(G_n)$ for some $d \leq c$. Let $x \in \tau_d(G_n)$. Thus there exists a positive integer m such that $x^m \in \gamma_d(G_n) \subseteq \tau_{c+1}(G_n)$. Since $\tau_{c+1}(G_n) \subseteq \tau_{d+1}(G_n)$ and $\tau_d(G_n)/\tau_{d+1}(G_n)$ is torsion-free, we obtain $x \in \tau_{d+1}(G_n)$ and so, $\tau_d(G_n) = \tau_{d+1}(G_n)$ which contradicts to residually torsion-free nilpotency. For positive integers i and c, with $1 \leq i \leq n$, let $x_{i,c} = x_i\tau_{c+1}(G_n)$. Thus $G_{n,c}$ is freely generated by the set $\{x_{i,c} : i = 1, \ldots, n\}$. As shown in Proposition 3.2, and the proof of Theorem 3, $\mathcal{L}_K(G_{n,c})$ is a relatively free nilpotent Lie algebra of rank n; freely generated by the set $\{\log x_{1,c}, \ldots, \log x_{n,c}\}$. Since $G_{n,c}$ has class c, we obtain $\mathcal{L}_K(G_{n,c})$ has class c (see [11, Theorem 7.3]).

The following result help us to construct the inverse limit of $\mathcal{L}_K(G_{n,c})$ with $c \geq 1$ and n a fixed positive integer with $n \geq 2$. The proof is based on some ideas of the proof given in Lemma 3.5. (See, also, Lemma 3.9.)

Lemma 3.12 *For positive integers c, d, with $c \leq d$, we write $\pi_{c,d}$ for the natural epimorphism from $G_{n,d}$ onto $G_{n,c}$ sending $g\tau_{d+1}(G_n)$ to $g\tau_{c+1}(G_n)$ for all $g \in G_n$. Then there exists a Lie algebra epimorphism $\xi_{\pi_{c,d}}$ from $\mathcal{L}_K(G_{n,d})$ onto $\mathcal{L}_K(G_{n,c})$ such that $\xi_{\pi_{c,d}}(\log(g\tau_{d+1}(G_n))) = \log(g\tau_{c+1}(G_n))$ for all $g \in G_n$. In particular, $\xi_{\pi_{c,d}}(\log x_{i,d}) = \log x_{i,c}$ for $i = 1, \ldots, n$, and $\xi_{\pi_{c,d}}(v(\log x_{1,d}, \ldots, \log x_{n,d})) = v(\log x_{1,c}, \ldots, \log x_{n,c})$ for all $v(\log x_{1,d}, \ldots, \log x_{n,d}) \in \mathcal{L}_K(G_{n,d})$.* $\square$

Form the following inverse limit

$$\widetilde{\mathcal{L}(G_n)} = \varprojlim(\mathcal{L}_K(G_{n,c}), \xi_{\pi_{c,d}}).$$

Throughout this paper we abbreviate $\widetilde{\mathcal{L}}_n = \widetilde{\mathcal{L}(G_n)}$, with $n \geq 2$. A typical element of $\widetilde{\mathcal{L}}_n$ has the form $(v_1, v_2, \ldots)$ where $v_c \in \mathcal{L}_K(G_{n,c})$, with $c \geq 1$, and $\xi_{\pi_{c,d}}(v_d) = v_c$ for $c \leq d$. Write, for $i = 1, \ldots, n$,

$$t_i = (\log(x_i\tau_2(G_n)), \log(x_i\tau_3(G_n)), \ldots, \log(x_i\tau_{c+1}(G_n)), \ldots) \in \widetilde{\mathcal{L}}_n.$$

Let $\mathcal{L}_n = \mathcal{L}(G_n)$ be the Lie subalgebra of $\widetilde{\mathcal{L}}_n$ generated by the set $\{t_1, \ldots, t_n\}$. Recall that for a positive integer c, $L_c(G_n) = \tau_c(G_n)/\tau_{c+1}(G_n)$. Since $L_c(G_n)$ is a free abelian group of finite rank ≥ 1, we may tensor it by K to obtain a vector space $L_{c,K}(G_n)$ over K. In particular, if X_c is any $\mathbb{Z}$-basis of $L_c(G_n)$ then every element of $L_{c,K}(G_n)$ may be uniquely written as a K-linear combination of elements $1 \otimes x$, with $x \in X_c$. Let $L_K(G_n) = \oplus_{c\geq 1} L_{c,K}(G_n)$. For a positive integer c, and a Lie algebra L, we write $\gamma_c(L)$ for the c-th term of the lower central series of L. Notice that $\gamma_d(L_K(G_n)) = \oplus_{i\geq d} L_{i,K}(G_n)$. $\square$

Theorem 4 *Let $\mathcal{L}_n$ be the Lie subalgebra of $\widetilde{\mathcal{L}}_n$ generated by the set $\{t_1, \ldots, t_n\}$. Then $\mathcal{L}_n$ is a relatively free Lie algebra freely generated by the set $\{t_1, \ldots, t_n\}$.*

Proof. For a positive integer n, with $n \geq 2$, let A_n be the free associative algebra over K; freely generated by the set $\{\ell_1, \ldots, \ell_n\}$. Give on A_n the structure of a Lie algebra by defining $[u, v] = uv - vu$ for all $u, v \in A_n$. Let L_n be the Lie subalgebra of A_n generated by the set $\{\ell_1, \ldots, \ell_n\}$. It is well-known that L_n is a free Lie algebra; freely generated by the set $\{\ell_1, \ldots, \ell_n\}$ (see, for example, [10]). Consider the natural epimorphism

$$\sigma_n : L_n \longrightarrow \mathcal{L}_n, \text{ with } \ell_i \mapsto t_i, \ i = 1, \ldots, n.$$

Then $L_n/\mathrm{ker}\sigma_n \cong \mathcal{L}_n$. Notice that $v(t_1, \ldots, t_n) = 0$ if and only if $v(\ell_1, \ldots, \ell_n) \in \mathrm{ker}\sigma_n$. To prove that $\mathcal{L}_n$ is relatively free, it is enough to show that $\mathrm{ker}\sigma_n$ is a verbal ideal (see [1, Chapter 14, page 275]). As for groups [16, Theorem 12.34, page 5] verbal ideals turn out to be precisely the fully invariant ideals of L_n (that is, those ideals which are invariant under all endomorphisms of L_n). Thus it is enough to show that $\mathrm{ker}\sigma_n$ is fully invariant i.e. if $v(\ell_1, \ldots, \ell_n) \in \mathrm{ker}\sigma_n$, then $v(v_1, \ldots, v_n) \in \mathrm{ker}\sigma_n$ for all $v_1, \ldots, v_n \in L_n$ or, equivalently, if $v(t_1, \ldots, t_n) = 0$, then $v(u_1, \ldots, u_n) = 0$ for all $u_1, \ldots, u_n \in \mathcal{L}_n$. Each t_i may be considered as a function on $\mathbb{N}$ such that $t_i(c) = \log x_{i,c} \in \mathcal{L}_K(G_{n,c})$ for all $c \in \mathbb{N}$. The element $v(t_1, \ldots, t_n)$ is considered as a function on $\mathbb{N}$. Thus $v(t_1(c), \ldots, t_n(c)) = 0$ for all $c \in \mathbb{N}$. Hence $v(t_1(c), \ldots, t_n(c)) = 0$ in each $\mathcal{L}_K(G_{n,c})$. For any positive integer c, $\mathcal{L}_K(G_{n,c})$ is a relatively free nilpotent Lie algebra; freely generated by the set $\{\log x_{1,c}, \ldots, \log x_{n,c}\}$. Since $t_i(c) = \log x_{i,c}$ for $i = 1, \ldots, n$, we obtain $\{t_1(c), \ldots, t_n(c)\}$ is a free generating set for $\mathcal{L}_K(G_{n,c})$. Therefore $v(w_{1,c}, \ldots, w_{n,c}) = 0$ for all $w_{1,c}, \ldots, w_{n,c} \in \mathcal{L}(G_{n,c})$. By the construction of $\widetilde{\mathcal{L}}_n$, we have $\widetilde{\mathcal{L}}_n \in \mathrm{var}(\{\mathcal{L}_K(G_{n,c}) : c \in \mathbb{N}\})$. But $v(\ell_1, \ldots, \ell_n)$ is an identity in the cartesian product $\prod_{c\geq 1} \mathcal{L}_K(G_{n,c})$. (The arguments given in the proof of 15.1 of [16, page 15]

can be adopted here without any changes.) Hence $v(\ell_1, \ldots, \ell_n)$ is an identity in $\widetilde{\mathcal{L}}_n$. Since $\mathcal{L}_n \subseteq \widetilde{\mathcal{L}}_n$, we obtain $v(\ell_1, \ldots, \ell_n)$ is an identity in $\mathcal{L}_n$. Therefore, $\mathcal{L}_n$ is a relatively free Lie algebra; freely generated by the set $\{t_1, \ldots, t_n\}$. $\square$

3.4 Verbal ideals

Let $\mathcal{F}_\infty$ be the free Lie algebra on a countably infinite set $\{\omega_1, \omega_2, \ldots, \}$. For each $n \geq 1$, the free Lie algebra $\mathcal{F}_n$ freely generated by $\omega_1, \ldots, \omega_n$ will then be embedded in $\mathcal{F}_\infty$, in a natural way. This free Lie algebra $\mathcal{F}_\infty$ is introduced for the special purpose to provide 'words': an element $w \in \mathcal{F}_\infty$ is called a *word* in the variables $\omega_1, \omega_2, \ldots$. Each word w involves only finitely many variables, in the sense that $w \in \mathcal{F}_n$ for some $n \in \mathbb{N}$. A set of words V is *closed* if it is, as subset of $\mathcal{F}_\infty$, a fully invariant ideal of $\mathcal{F}_\infty$. In particular, if $v \in V$ is a word in n variables, and $(u_1, \ldots, u_n) \in \mathcal{F}_\infty^n$ an n-tuple of words, then $v(u_1, \ldots, u_n) \in V$. The *closure* of an arbitrary set $\mathfrak{w}$ of words is defined as the intersection of all closed sets containing $\mathfrak{w}$. (Since the set of all words is a closed set containing $\mathfrak{w}$, the aforementioned definition makes sense.) Let Y be a set of words involving $\omega_1, \ldots, \omega_n$ only such that Y is a fully invariant ideal of $\mathcal{F}_n$ (hence it is a verbal ideal). If V denotes the closure of Y then $V \cap \mathcal{F}_n = Y$ (see, for example, [16, page 7]. The arguments given in page 7 can be adopted here without any changes]). That is, the closure of a fully invariant ideal of $\mathcal{F}_n$ intersected with $\mathcal{F}_n$ leads back to the original ideal. Let L_∞ be a free Lie algebra on a countably infinite set $\{\ell_1, \ell_2, \ldots\}$. For a positive integer n, with $n \geq 2$, let L_n denote the free Lie algebra freely generated by the set $\{\ell_1, \ldots, \ell_n\}$. Let ξ be the natural mapping of $\mathcal{F}_\infty$ onto L_∞; it is given by $\xi(\omega_i) = \ell_i$ for $i \geq 1$. It is an isomorphism between the two free Lie algebras which, restricted to $\mathcal{F}_n \subset \mathcal{F}_\infty$ maps $\mathcal{F}_n$ onto $L_n \subset L_\infty$. So we have the following diagram:

$$\begin{tikzcd}
\mathcal{F}_\infty \arrow[r] \arrow[d, "\xi"'] & \mathcal{F}_n \arrow[d, dotted] & \\
L_\infty \arrow[r] & L_n \arrow[r, "\sigma_n"] & \mathcal{L}_n
\end{tikzcd}$$

Let V be a fully invariant ideal of $\mathcal{F}_\infty$. The words in V that involve only $\omega_1, \ldots, \omega_n$ are given by $V \cap \mathcal{F}_n$. Of course, $V \cap \mathcal{F}_n$ is a fully invariant ideal of $\mathcal{F}_n$. For a positive integer n, with $n \geq 2$, let V_n be the fully invariant ideal of $\mathcal{F}_n$ such that $\xi(V_n) = \ker\sigma_n$. Let Y_n be the closure of V_n. Then $Y_n \cap \mathcal{F}_n = V_n$. Since ξ is bijective, we obtain $\xi(Y_n) \cap L_n = \xi(V_n) = \ker\sigma_n$. Recall

that there is a $1-1$ correspondence between the varieties $\mathfrak{V}$ of Lie algebras and the fully invariant ideals V of $\mathcal{F}_\infty$ (or the closed sets of words V or the verbal ideals V). Let $\mathfrak{V}_n$ (or simply $\mathfrak{V}$) be the variety of Lie algebras corresponding to Y_n. Write $\mathfrak{V}(L_n) = \xi(Y_n) \cap L_n$. That is, the verbal ideal of L_n corresponding to $\mathfrak{V}$. Notice that $\mathfrak{V}(L_n) = \ker\sigma_n$. It is well known that $L_n = \oplus_{m\geq 1} L_n^m$, where L_n^m denotes the vector subspace of L_n spanned by all Lie commutators $[\ell_{i_1}, \ldots, \ell_{i_m}]$ with $i_1, \ldots, i_m \in \{1, \ldots, n\}$. Since K is an infinite field, we obtain $\mathfrak{V}$ is (multi-)homogeneous. That is, given an identity $v \equiv 0$ in $\mathfrak{V}$ such that $v = \sum_\alpha v_\alpha$ is the (multi-)homogeneous decomposition of v, then for all α, $v_\alpha \equiv 0$ is also an identity in $\mathfrak{V}$. The proof of the following result is elementary.

Lemma 3.13 *For a positive integer n, with $n \geq 2$, let L_n be a free Lie algebra of rank n. Let I be a proper fully invariant ideal of L_n. Then $I \subseteq L_n'$.* $\square$

Since $\ker\sigma_n \subseteq L_n'$ (by the proof of Theorem 4 and Lemma 3.13),

$$\ker\sigma_n = \mathfrak{V}(L_n) = \oplus_{m\geq 2}(\ker\sigma_n \cap L_n^m).$$

Since $\mathfrak{V}(L_n) = \oplus_{m\geq 2}(\mathfrak{V}(L_n)\cap L_n^m)$, it is easily verified that $\mathcal{L}_n = \oplus_{m\geq 1}\mathcal{L}_n^m$, where $\mathcal{L}_n^m = (L_n^m + \mathfrak{V}(L_n))/\mathfrak{V}(L_n)$. Furthermore, $\gamma_c(\mathcal{L}_n) = \oplus_{m\geq c}\mathcal{L}_n^m$ for all c. For positive integers i and c, with $i \leq c$, let n_i denote the rank of the free abelian group $L_i(G_{n,c})$ ($= \gamma_i(G_{n,c})/\gamma_{i+1}(G_{n,c})$). Furthermore, let $f(i) = n_1 + \cdots + n_{i-1}$, with $n_0 = 0$ and $n_1 = n$. Let $B_{i,c} = \{a_{f(i)+1,c}, \ldots, a_{f(i+1,c)}\}$ be a subset of $\gamma_i(G_{n,c})$ such that the set $\{a_{f(i)+1,c}\gamma_{i+1}(G_{n,c}), \ldots, a_{f(i+1),c}\gamma_{i+1}(G_{n,c})\}$ is a $\mathbb{Z}$-basis of $L_i(G_{n,c})$. Thus $B_c = \cup B_{i,c} = \{a_{1,c}, \ldots, a_{f(c+1),c}\}$ is a canonical basis of $G_{n,c}$. For any positive integer c, we choose a canonical basis B_c of $G_{n,c}$ subject to $a_{\kappa,c} = x_{\kappa,c}$, $\kappa = 1, \ldots, n$. For a positive integer c, with $c \geq 2$, let $\pi_{c-1,c}$ be the natural epimorphism from $G_{n,c}$ onto $G_{n,c-1}$ sending $g\tau_{c+1}(G_n)$ to $g\tau_c(G_n)$ for all $g \in G_n$. Then $\ker\pi_{c-1,c} = \tau_c(G_n)/\tau_{c+1}(G_n)$. By Lemma 3.12, there exists a Lie algebra epimorphism $\xi_{\pi_{c-1,c}}$ from $\mathcal{L}_K(G_{n,c})$ onto $\mathcal{L}_K(G_{n,c-1})$ such that $\xi_{\pi_{c-1,c}}(\log(g\tau_{c+1}(G_n))) = \log(g\tau_c(G_n))$ for all $g \in G_n$. In particular, $\xi_{\pi_{c-1,c}}(\log x_{i,c}) = \log x_{i,c-1}$ for $i = 1, \ldots, n$.

Proposition 3.4 *For a positive integer n, with $n \geq 2$, let H_n be a relatively free group of rank n. Then $\tau_c(H_n) = \gamma_c(H_n)\tau_{c+1}(H_n)$ for all c.*

Proof. Let H_n be a relatively free group of rank n, with $n \geq 2$ freely generated by the set $\mathfrak{h}$. The commutator group H_n/H_n' is an abelian relatively free group freely generated by $\mathfrak{h}$ taken

modulo H_n'. Suppose first that H_n/H_n' is a free abelian group of exponent m, with $m > 0$. Then $\tau(H_n/H_n') = H_n/H_n'$. Since $\tau_2(H_n)/\gamma_2(H_n) = \tau(H_n/H_n')$, we obtain $\tau_2(H_n) = H_n$. We claim that $H_n = \tau_c(H_n)$ for all c. Since $H_n = \tau_2(H_n)$, we have $x \in \tau_2(H_n)$ for all $x \in H_n$. Since H_n/H_n' has exponent m, we get $x^m \in H_n'$. Then $(x^m, y) \in \gamma_3(H_n)$ for all $y \in H_n$. Using repeatedly the commutator identity $(ab, c) = (a, c)(a, c, b)(b, c)$, we obtain $(x, y)^m \in \gamma_3(H_n)$ for all $y \in H_n$. So, $(x, y) \in \tau_3(H_n)$ for all $y \in H_n$. It is clearly enough that $(x, y) \in \tau_3(H_n)$ for all $x, y \in H_n$. Since $\tau_3(H_n)$ is fully invariant, we have $\gamma_2(H_n) \subseteq \tau_3(H_n)$. Let $x \in \tau_2(H_n)$. Then $x^m \in \gamma_2(H_n)$ and so $x^m \in \tau_3(H_n)$. Since $\tau_2(H_n)/\tau_3(H_n)$ is torsion-free, we get $x \in \tau_3(H_n)$. Therefore $\tau_2(H_n) = \tau_3(H_n)$. Continuing this process we obtain $H_n = \tau_c(H_n)$ for all c. So, we get the required result.

Thus we may assume that H_n/H_n' is a free abelian group with a basis (i.e. a free generating set) $\mathfrak{h}$ taken modulo H_n'. Assume that there are no repetitions of terms of the series $\{\tau_c(H_n)\}_{c\geq 1}$. Fix a positive integer c. It is clearly enough that we may assume that $c \geq 2$. It is enough to show that $\tau_c(H_n) \subseteq \gamma_c(H_n)\tau_{c+1}(H_n)$. Write $H_{n,c} = H_n/\tau_{c+1}(H_n)$. Notice that $H_{n,c}$ is a relatively free nilpotent torsion-free group of rank n and class c. (If the class of $H_{n,c}$ is strictly less than c, then it be can easily shown that there are repetitions of the series $\{\tau_c(H_n)\}_{c\geq 1}$.) By Theorem A (I), $H_{n,c}$ is Magnus and so, $\tau_\kappa(H_{n,c}) = \gamma_\kappa(H_{n,c})$ for all κ, $\kappa \in \{1, \ldots, c\}$. Write $E_{n,c} = H_n/\gamma_c(H_n)\tau_{c+1}(H_n)$. Since $\gamma_c(H_{n,c}) = \gamma_c(H_n)\tau_{c+1}(H_n)/\tau_{c+1}(H_n)$, we have $H_{n,c}/\gamma_c(H_{n,c}) \cong E_{n,c}$. Thus $E_{n,c}$ is torsion-free. Let $\rho_{c-1,c}$ denote the natural epimorphism from $E_{n,c}$ onto $H_{n,c-1}$ sending $h(\gamma_c(H_n)\tau_{c+1}(H_n))$ to $h\tau_c(H_n)$ for all $h \in H_n$. It is easily verified that $\ker\rho_{c-1,c} = \tau_c(H_n)/\gamma_c(H_n)\tau_{c+1}(H_n)$. Furthermore, it is easy to see that $\ker\rho_{c-1,c}$ is an homomorphic image of $\tau_c(H_n)/\gamma_c(H_n)$. Since $H_n/\gamma_c(H_n)$ is a finitely generated nilpotent group, we have $\tau(H_n/\gamma_c(H_n))$ is a finite group and so, $\tau_c(H_n)/\gamma_c(H_n)$ is a finite group. Therefore $\ker\rho_{c-1,c}$ is finite. Since $E_{n,c}$ is torsion-free, we obtain $\ker\rho_{c-1,c}$ is trivial and so $\tau_c(H_n) \subseteq \gamma_c(H_n)\tau_{c+1}(H_n)$. Hence $\tau_c(H_n) = \gamma_c(H_n)\tau_{c+1}(H_n)$ for all c. Finally we assume that there are repetitions of terms of the series $\{\tau_c(H_n)\}_{c\geq 1}$. Since H_n/H_n' is free abelian, we have $\tau_2(H_n) = H_n'$. Let κ be the smallest positive integer such that $\tau_\kappa(H_n) = \tau_{\kappa+1}(H_n)$ and $H_n \supset \tau_2(H_n) \supset \cdots \supset \tau_{\kappa-1}(H_n) \supset \tau_\kappa(H_n)$. Since $\tau_\kappa(H_n) = \tau_{\kappa+1}(H_n)$, we get $\tau_j(H_n) = \tau_\kappa(H_n)$ for all $j \geq \kappa + 1$. Hence our claim holds for all $j \geq \kappa$. If $\kappa = 2$ then $\tau_c(H_n) = \gamma_c(H_n)\tau_{c+1}(H_n)$ for all c. Thus we concentrate on $j < \kappa$ and $\kappa \geq 3$. Write $H_{n,c} = H_n/\tau_{c+1}(H_n)$ for $c = 1, \ldots, \kappa - 1$, and fix c. Using similar arguments as before we

obtain the required result. Hence, in any case, we have $\tau_c(H_n) = \gamma_c(H_n)\tau_{c+1}(H_n)$. $\square$

Remark 3.1 The proof given in the first part of the proof of Proposition 3.4 is independent up to the rank of H_n.

We deduce the following result.

Corollary 3.2 *Let $\mathfrak{L}$ be a residually torsion-free nilpotent variety of groups. For positive integers n and c, with $n \geq 2$, let $G_n = F_n(\mathfrak{L})$ be the relatively free group of rank n in $\mathfrak{L}$ and $G_{n,c} = G_n/\tau_{c+1}(G_n)$. Then, for all c, $\gamma_c(G_{n,c}) = \tau_c(G_n)/\tau_{c+1}(G_n)$.*

Proof. Since $\gamma_c(G_{n,c}) = \gamma_c(G_n)\tau_{c+1}(G_n)/\tau_{c+1}(G_n)$, we obtain from Proposition 3.4 the required result. $\square$

For a positive integer c, let ψ_c be the mapping from $\{\ell_1, \ldots, \ell_n\}$ into $\mathcal{L}_K(G_{n,c})$ such that $\psi_c(\ell_i) = t_i(c) = \log x_{i,c}$, $i = 1, \ldots, n$. Since L_n is free on $\{\ell_1, \ldots, \ell_n\}$, ψ_c extends to a Lie algebra homomorphism from L_n into $\mathcal{L}_K(G_{n,c})$. Since $\mathcal{L}_K(G_{n,c})$ is generated by the set $\{\log x_{1,c}, \ldots, \log x_{n,c}\}$, we have ψ_c is surjective. Let $v(\ell_1, \ldots, \ell_n) \in \ker\sigma_n$. Then $v(t_1, \ldots, t_n) = 0$ in $\mathcal{L}_n$ and so, $v(\log x_{1,c}, \ldots, \log x_{n,c}) = 0$ for all $c \in \mathbb{N}$ (see the proof of Theorem 4). Hence $\psi_c(v(\ell_1, \ldots, \ell_n)) = 0$ i.e. $v(\ell_1, \ldots, \ell_n) \in \ker\psi_c$. Therefore $\ker\sigma_n \subseteq \ker\psi_c$ and so, ψ_c induces a Lie algebra epimorphism $\widetilde{\psi}_c$, say, from $\mathcal{L}_n$ onto $\mathcal{L}_K(G_{n,c})$ (for all c) sending t_i to $\log x_{i,c}$ for $i = 1, \ldots, n$. So

$$\begin{tikzcd} L_n \arrow[r, "\sigma_n"] \arrow[dr, "\psi_c"'] & \mathcal{L}_n \arrow[d, dotted, "\widetilde{\psi}_c"] \\ & \mathcal{L}_K(G_{n,c}) \end{tikzcd}$$

Lemma 3.14 *For a positive integer c, $\widetilde{\psi}_c(\mathcal{L}_n^c) = \mathcal{L}_K(\gamma_c(G_{n,c}))$. In particular, for $c \geq 2$, $\widetilde{\psi}_c(\gamma_c(\mathcal{L}_n)) = \ker\xi_{\pi_{c-1,c}}$. Furthermore, $\ker\sigma_n = \cap_{c\geq 1}\ker\psi_c$.*

Proof. Recall that, for a positive integer c, $\mathcal{L}_n^c$ is the vector K-space spanned by all Lie commutators $[t_{i_1}, \ldots, t_{i_c}]$ with $i_1, \ldots, i_c \in \{1, \ldots, n\}$. Observe that

$$[t_{i_1}, \ldots, t_{i_c}] = (\underbrace{0, \ldots, 0}_{c-1}, [t_{i_1}(c), \ldots, t_{i_c}(c)], [t_{i_1}(c+1), \ldots, t_{i_c}(c+1)], \ldots) \in \widetilde{\mathcal{L}}_n.$$

Applying $\widetilde{\psi}_c$ on $[t_{i_1}, \ldots, t_{i_c}]$, we obtain $\widetilde{\psi}_c([t_{i_1}, \ldots, t_{i_c}]) = [t_{i_1}(c), \ldots, t_{i_c}(c)]$. Thus $\widetilde{\psi}_c(\mathcal{L}_n^c) \subseteq \gamma_c(\mathcal{L}(G_{n,c}))$. By Lemma 3.6 and since $G_{n,c}$ is Magnus, we have $\widetilde{\psi}_c(\mathcal{L}_n^c) \subseteq \mathcal{L}(\gamma_c(G_{n,c}))$. Let $c = 1$. Since $\ker\sigma_n \subseteq L_n'$, we obtain $\mathcal{L}_n/\mathcal{L}_n' \cong L_n/L_n'$ as vector spaces. Since $\mathcal{L}_n' = \oplus_{c\geq 2}\mathcal{L}_n^c$, we get $\dim \mathcal{L}_n^1 = n$ and so $\{t_1, \ldots, t_n\}$ is a K-basis for $\mathcal{L}_n^1$. Since $\dim \mathcal{L}(G_{n,1}) = \mathcal{H}(G_{n,1}) = n$, we obtain $\{t_1(c), \ldots, t_n(c)\}$ is a K-basis for $\mathcal{L}_K(G_{n,1})$. It is clearly enough that $\widetilde{\psi}_1(\mathcal{L}_n^1) = \mathcal{L}_K(G_{n,1})$. Thus we may assume that $c \geq 2$. By Lemma 3.5, $\mathcal{L}_K(\gamma_c(G_{n,c})) = \ker\xi_{\pi_{c-1,c}}$ and so, $\widetilde{\psi}_c(\mathcal{L}_n^c) \subseteq \ker\xi_{\pi_{c-1,c}}$. To prove that $\gamma_c(\mathcal{L}(G_{n,c})) \subseteq \widetilde{\psi}_c(\mathcal{L}_n^c)$ it is enough to show that $[t_{i_1}(c), \ldots, t_{i_c}(c)] \in \widetilde{\psi}_c(\mathcal{L}_n^c)$ for all $i_1, \ldots, i_c \in \{1, \ldots, n\}$. Since $\gamma_c(\mathcal{L}_K(G_{n,c})) = \mathcal{L}_K(\gamma_c(G_{n,c})) = \ker\xi_{\pi_{c-1,c}}$, we obtain $[t_{i_1}(c-1), \ldots, t_{i_c}(c-1)] = 0$. Consider the element

$$v = (\underbrace{0, \ldots, 0}_{c-1}, [t_{i_1}(c), \ldots, t_{i_c}(c)], [t_{i_1}(c+1), \ldots, t_{i_c}(c+1)], \ldots) \in \widetilde{\mathcal{L}}_n.$$

It is easily seen that $[t_{i_1}, \ldots, t_{i_c}] = v$, and $\widetilde{\psi}_c(v) = [t_{i_1}(c), \ldots, t_{i_c}(c)]$. Therefore $\widetilde{\psi}_c(\mathcal{L}_n^c) = \gamma_c(\mathcal{L}_K(G_{n,c})) = \ker\xi_{\pi_{c-1,c}}$. Since $\gamma_c(\mathcal{L}_n) = \oplus_{m\geq c}\mathcal{L}_n^m$, it is easily seen that $\widetilde{\psi}_c(\gamma_c(\mathcal{L}_n)) = \ker\xi_{\pi_{c-1,c}}$ for all $c \geq 2$.

Since $\ker\sigma_n \subseteq \ker\psi_c$ for all c, we have $\ker\sigma_n \subseteq \cap_{c\geq 1}\ker\psi_c$. Let $u = u(\ell_1, \ldots, \ell_n) \in \cap_{c\geq 1}\ker\psi_c$. Since $u \in \ker\psi_c$ for all c, we have $\psi_c(u(\ell_1, \ldots, \ell_n)) = u(t_1(c), \ldots, t_n(c)) = 0$ for all c. Since $\mathcal{L}_K(G_{n,c})$ is relatively free on $\{t_1(c), \ldots, t_n(c)\}$, we have $u(\ell_1, \ldots, \ell_n)$ is an identity for $\mathcal{L}_K(G_{n,c})$ for all c. This means that $u(\ell_1, \ldots, \ell_n)$ is an identity of the cartesian product $\prod_{c\geq 1}\mathcal{L}_K(G_{n,c})$. Thus, $u(\ell_1, \ldots, \ell_n)$ is an identity of $\widetilde{\mathcal{L}}_n$ and so, it is an identity of $\mathcal{L}_n$. That is, $u(\ell_1, \ldots, \ell_n) \in \mathfrak{V}(L_n) = \ker\sigma_n$. Therefore $\ker\sigma_n = \cap_{c\geq 1}\ker\psi_c$. $\Box$

In the next few lines, we write $I_c = \ker\widetilde{\psi}_c$. We claim that I_c is a fully invariant ideal of $\mathcal{L}_n$. It is enough to show that $\vartheta(I_c) \subseteq I_c$ for any ϑ endomorphism of $\mathcal{L}_n$. Let $u = u(t_1, \ldots, t_n) \in I_c$. Then $\widetilde{\psi}_c(u(t_1, \ldots, t_n)) = u(\log x_{1,c}, \ldots, \log x_{n,c}) = 0$ in $\mathcal{L}_K(G_{n,c})$. Since $\mathcal{L}_K(G_{n,c})$ is relatively free on $\{\log x_{1,c}, \ldots, \log x_{n,c}\}$, we obtain $u(u_1, \ldots, u_n) = 0$ for all $u_1, \ldots, u_n \in \mathcal{L}_K(G_{n,c})$. Let ϑ be an endomorphism of $\mathcal{L}_n$. Then $\vartheta(t_i) = v_i$ for $i = 1, \ldots, n$. Since $\widetilde{\psi}_c$ is a Lie algebra homomorphism from $\mathcal{L}_n$ into $\mathcal{L}(G_{n,c})$, $\widetilde{\psi}_c(v_i) = w_i$ for $i = 1, \ldots, n$. But $u(w_1, \ldots, w_n) = 0$ (in $\mathcal{L}_K(G_{n,c})$) and so, $\widetilde{\psi}_c(\vartheta(u(t_1, \ldots, t_n))) = \widetilde{\psi}_c(u(v_1, \ldots, v_n)) = u(w_1, \ldots, w_n) = 0$. Therefore $\vartheta(u(t_1, \ldots, t_n)) \in I_c$. Hence I_c is fully invariant (and so, I_c is verbal). Since $\mathcal{L}_n' = \oplus_{m\geq 2}\mathcal{L}_n^m$ and I_c is a proper fully invariant ideal of $\mathcal{L}_n$, it is easy to see that $I_c \subseteq \mathcal{L}_n'$.

Next we claim that $I_c = \gamma_{c+1}(\mathcal{L}_n)$. Since $\mathcal{L}_n/I_c \cong \mathcal{L}_K(G_{n,c})$ and $\mathcal{L}_K(G_{n,c})$ has class c, it is enough to show that $I_c \subseteq \gamma_{c+1}(\mathcal{L}_n)$. To get a contradiction, let $w \in I_c$ but not in $\gamma_{c+1}(\mathcal{L}_n)$. Since I_c is a proper fully invariant ideal and since K is an infinite field, we may assume that $w \in I_c \cap \mathcal{L}_n^d$ for some d, with $2 \le d \le c$. We write

$$w = w(t_1, \ldots, t_n) = \sum \alpha_{(i_1,\ldots,i_d)}[t_{i_1}, \ldots, t_{i_d}],$$

$\alpha_{(i_1,\ldots,i_d)} \in K$. Since $\widetilde{\psi}_c(w) = 0$, we obtain

$$w(t_1(c), \ldots, t_n(c)) = \sum \alpha_{(i_1,\ldots,i_d)}[t_{i_1}(c), \ldots, t_{i_d}(c)] = 0$$

in $\mathcal{L}_K(G_{n,c})$. Making use of $\xi_{\pi_{\mu,c}}$ (Lemma 3.12), with $\mu \le c$, we have $w(t_1(\mu), \ldots, t_n(\mu)) = 0$ for all $\mu \le c$. Notice that

$$\widetilde{\psi}_\kappa(w) = w(t_1(\kappa), \ldots, t_n(\kappa)) \in \gamma_d(\mathcal{L}(G_{n,\kappa}))$$

for all $\kappa \ge c$. Since $\xi_{\pi_{c,c+1}}(w(t_1(c+1)), \ldots, t_n(c+1)) = w(t_1(c), \ldots, t_n(c)) = 0$, we obtain $w(t_1(c+1), \ldots, t_n(c+1)) \in \ker\xi_{\pi_{c,c+1}}$. By Lemma 3.5 and (Lemma 3.6), $w(t_1(c+1), \ldots, t_n(c+1)) \in \gamma_{c+1}(\mathcal{L}(G_{n,c+1}))$. For a positive integer m, with $m = 1, \ldots, c$, let $\mathcal{L}_m(G_{n,c})$ denote the vector subspace of $\mathcal{L}_K(G_{n,c})$ spanned by all Lie commutators of the form $[\log x_{i_1,c}, \ldots, \log x_{i_m,c}]$. By Theorem 3 and since $L_K(G_{n,c})$ is graded,

$$\mathcal{L}_K(G_{n,c}) = \oplus_{m=1}^{c} \mathcal{L}_m(G_{n,c}).$$

Since $w(t_1(c+1), \ldots, t_n(c+1))$ has length d and $d \le c$, we obtain $w(t_1(c+1), \ldots, t_n(c+1)) = 0$ (in $\mathcal{L}_K(G_{n,c+1})$). Thus $\widetilde{\psi}_{c+1}(w) = 0$. Continue this process, we finally obtain $w = 0$ which is a contradiction. Therefore $I_c = \gamma_{c+1}(\mathcal{L}_n)$. Furthermore, $\mathcal{L}_n^c \cong \gamma_c(\mathcal{L}_n)/\gamma_{c+1}(\mathcal{L}_n) \cong \gamma_c(\mathcal{L}_K(G_{n,c}) = \mathcal{L}_K(\gamma_c(G_{n,c}))$ (by Lemma 3.6). Thus we obtain the following result.

Proposition 3.5 *For a positive integer c, $\mathcal{L}_n/\gamma_{c+1}(\mathcal{L}_n) \cong \mathcal{L}_K(G_{n,c})$ as Lie algebras under the isomorphism φ_{c+1} sending $t_i + \gamma_{c+1}(\mathcal{L}_n)$ to $\log x_{i,c}$, $i = 1, \ldots, n$. In particular, $\varphi_{c+1}(u(t_1, \ldots, t_n) + \gamma_{c+1}(\mathcal{L}_n)) = u(\log x_{1,c}, \ldots, \log x_{n,c})$ for all $u(t_1, \ldots, t_n) \in \mathcal{L}_n$ and $c \ge 1$. Furthermore, for all positive integers c, $\mathcal{L}_n^c \cong \mathcal{L}_K(\gamma_c(G_{n,c}))$ as vector spaces via the linear isomorphism $\varphi_{c+1}\eta_c$, where η_c is the natural linear isomorphism from $\mathcal{L}_n^c$ onto $\gamma_c(\mathcal{L}_n)/\gamma_{c+1}(\mathcal{L}_n)$ sending u to $u + \gamma_{c+1}(\mathcal{L}_n)$.* $\square$

Since $G_n = F_n(\mathfrak{L}) = F_n/\mathfrak{L}(F_n)$ is relatively free; freely generated by the set $\{x_1, \ldots, x_n\}$, we obtain $G_n/G_n' \cong F_n/\mathfrak{L}(F_n)F_n'$ is an abelian relatively free group freely generated by the set $\{x_1G_n', \ldots,\ x_nG_n'\}$. Thus G_n/G_n' is either free abelian with basis $\{x_1G_n', \ldots, x_nG_n'\}$ and so, $F_n/\mathfrak{L}(F_n)F_n'$ has exponent zero, that is, $\mathfrak{L}(F_n) \subseteq F_n'$ and G_n satisfies commutator laws, or it is the abelian group of exponent m every element of whose basis $\{x_1G_n', \ldots, x_nG_n'\}$ has order m for some $m > 0$. If $F_n/\mathfrak{L}(F_n)F_n'$ has exponent m then $x^m \in \mathfrak{L}$. Since G_n is torsion-free, we obtain G_n/G_n' is a free abelian of rank n with basis $\{x_1G_n', \ldots, x_nG_n'\}$. Thus $\tau_2(G_n) = G_n'$. Our next result is about relative freeness of $L_K(G_n)$.

Theorem 5 *The Lie algebra $\mathcal{L}_n$ is isomorphic to $L_K(G_n)$ via a Lie algebra isomorphism sending t_i to x_iG_n', $i = 1, \ldots, n$. In particular, $L_K(G_n)$ is relatively free Lie algebra; freely generated by the set $\{x_1G_n', \ldots, x_nG_n'\}$.*

Proof. Let σ_n be the natural mapping from L_n onto $\mathcal{L}_n$ sending ℓ_i to t_i, $i = 1, \ldots, n$. By the proof of Theorem 4, we obtain $\ker\sigma_n$ is a fully invariant ideal of L_n. Let $v \in \ker\sigma_n \cap L_n^m$ for some $m \geq 2$, and write v as a linear combination of Lie commutators $[\ell_{i_1}, \ldots, \ell_{i_m}]$ i.e. $v = \sum \alpha_{(i_1,\ldots,i_m)}[\ell_{i_1}, \ldots, \ell_{i_m}]$. Thus $\sigma_n(v) = \sum \alpha_{(i_1,\ldots,i_m)}[t_{i_1}, \ldots, t_{i_m}] = 0$. Therefore, for all c, $\sum \alpha_{(i_1,\ldots,i_m)}[t_{i_1}(c), \ldots, t_{i_m}(c)] = 0$ (as in proof of Theorem 4). That is,

$$\sum \alpha_{(i_1,\ldots,i_m)}[\log x_{i_1,c}, \ldots, \log x_{i_m,c}] = 0 \tag{15}$$

in $\mathcal{L}_K(G_{n,c})$. By the BCH formula, we obtain

$$[\log x_{i_1,c}, \ldots, \log x_{i_m,c}] = \log(x_{i_1,c}, \ldots, x_{i_m,c}) + u,$$

where $u \in \mathcal{L}_K(\gamma_{m+1}(G_{n,c}))$, and so, we have from (15)

$$\sum \alpha_{(i_1,\ldots,i_m)} \log(x_{i_1,c}, \ldots, x_{i_m,c}) \in \mathcal{L}_K(\gamma_{m+1}(G_{n,c})).$$

Using similar arguments as in the proof of Theorem 3, we obtain $\sum \alpha_{(i_1,\ldots,i_m)}[x_{i_1,c}G_{n,c}', \ldots, x_{i_m,c}G_{n,c}'] = 0$ in $L_K(G_{n,c})$ for all c. But $\mathcal{L}_K(G_{n,c})$ is naturally isomorphic to $L_K(G_{n,c})$. By Theorem A (I), $\mathcal{L}_K(G_{n,c})$ is a relatively free nilpotent Lie algebra and so is $L_K(G_{n,c})$. So, v is an identity in each $L_K(G_{n,c})$. Hence, v is an identity in the cartesian product $L_K(G_{n,1}) \times L_K(G_{n,2}) \times \cdots$ therefore, v is an identity in $\varprojlim L_K(G_{n,c})$. Since $L_K(G_{n,c}) \cong L_K(G_n)/\gamma_{c+1}(L_K(G_n))$, as Lie algebras, in a natural way, it is easily verified that the

$\varprojlim L_K(G_{n,c})$ is isomorphic, as a Lie algebra, to $\varprojlim L_K(G_n)/\gamma_{c+1}(L_K(G_n))$. Thus v is an identity in $\varprojlim L_K(G_n)/\gamma_{c+1}(L_K(G_n))$. Since $L_K(G_n)$ is embedded into $\varprojlim L_K(G_n)/\gamma_{c+1}(L_K(G_n))$ in a natural way, we have v is an identity in $L_K(G_n)$. Hence $L_K(G_n) \in \mathfrak{V}$. By Proposition 3.4 and Lemma 3.2, $L_K(G_n)$ is generated by the set $\{x_1G'_n, \ldots, x_nG'_n\}$. Let φ be the natural epimorphism from $\mathcal{L}_n$ into $L_K(G_n)$ sending t_i to $x_iG'_n$, $i = 1, \ldots, n$. We claim that φ is one-to-one. By Proposition 3.5, $\mathcal{L}_n/\gamma_{c+1}(\mathcal{L}_n)$ is isomorphic as a Lie algebra to $\mathcal{L}_K(G_{n,c})$ via an isomorphism φ_{c+1} for all c. Thus

$$\begin{aligned} \mathcal{L}_n/\gamma_{c+1}(\mathcal{L}_n) &\overset{\varphi_{c+1}}{\cong} \mathcal{L}_K(G_{n,c}) \\ &\cong L_K(G_{n,c}) \\ &\cong L_K(G_n)/\gamma_{c+1}(L_K(G_n)) \end{aligned}$$

for all c. Since $\mathcal{L}_n$ is residually nilpotent, i.e. $\cap_{c\geq 1}\gamma_{c+1}(\mathcal{L}_n) = \{0\}$, and for all c, $\mathcal{L}_n/\gamma_{c+1}(\mathcal{L}_n) \cong L_K(G_n)/\gamma_{c+1}(L_K(G_n))$, we obtain $\mathcal{L}_n$ is isomorphic as a Lie algebra to $L_K(G_n)$. By Theorem 4, we obtain the Lie algebra $L_K(G_n)$ is relatively free. It is easily verified that the set $\{x_1G'_n, \ldots, x_nG'_n\}$ freely generates $L_K(G_n)$. $\square$

4 Proofs of Theorems A and B

4.1 Proof of Theorem A

(I) It follows from Proposition 3.2 and Theorem 3 that $\mathcal{L}_K(F_n(\mathfrak{T}_c))$ is relatively free in some variety of nilpotent Lie algebras, and $\mathcal{L}_K(F_n(\mathfrak{T}_c)) \cong L_K(F_n(\mathfrak{T}_c))$ as Lie algebras in a natural way. Write $G = F_n(\mathfrak{T}_c)$, with $n \geq 2$, freely generated by the set $\{y_1, \ldots, y_n\}$. From the proof of Theorem 3, G/G' is free abelian group of rank n, and so, $\tau_2(G) = G'$. Recall that $L^{(S)}(G) = \oplus_{1\leq i\leq c} L_i^{(S)}(G)$, where $L_i^{(S)}(G) = \gamma_i(G)\tau_{i+1}(G)/\tau_{i+1}(G)$, $i = 1, \ldots, c$. The additive group of $L^{(S)}(G)$ is free abelian, $L_{\mathbb{Q}}(G) = \mathbb{Q} \otimes_{\mathbb{Z}} L^{(S)}(G)$ and $L^{(S)}(G)$ is a subset of $L_{\mathbb{Q}}(G)$. By Lemma 3.2, $L^{(S)}(G)$ is generated as a Lie ring by the set $\{y_1G', \ldots, y_nG'\}$. We give on $L_{\mathbb{Q}}(G)$ the structure of a group by means of BCH formula, denoted by R. (Notice that $R = L_{\mathbb{Q}}(G)$ as sets.) By Lemma 3.3, $\{y_1G' + L_{\mathbb{Q}}(G)', \ldots, y_nG' + L_{\mathbb{Q}}(G)'\}$ is a $\mathbb{Q}$-basis for $L_{\mathbb{Q}}(G)/L_{\mathbb{Q}}(G)'$. Let H be the subgroup of R generated by the set $\{y_1G', \ldots, y_nG'\}$. (Notice that the identity element of H is the zero element in $L_{\mathbb{Q}}(G)$.) By Lemma 2.1, H is a torsion-free finitely generated nilpotent group of class c and $\tau_2(H) = H'$.

Next we shall show that there is a group isomorphism

$$\gamma_i(H)/\gamma_{i+1}(H) \cong \gamma_i(L^{(S)}(G))/\gamma_{i+1}(L^{(S)}(G)) \tag{16}$$

for all i, with $i = 1, \ldots, c$. (For the proof of the aforementioned isomorphism, we use some arguments given in [19].) First we shall show that $\gamma_c(H) = \gamma_c(L^{(S)}(G))$. Every element of $\gamma_c(H)$ can be written as a product of group commutators (in the sense of the operation $\circ$) of length c in the $y_1G', \ldots, y_nG'$. By the BCH formula, we may deduce that every group commutator of length c in the $y_1G', \ldots, y_nG'$ lies in $L_{c,\mathbb{Q}}(G)$ $(= \mathbb{Q} \otimes_{\mathbb{Z}} \gamma_c(G))$. Since $\gamma_{c+1}(L^{(S)}(G)) = \{0\}$, the multiplication of group commutators of length c in H is equal to their addition in the ring $L^{(S)}(G)$. Hence it follows that $\gamma_c(H) \subseteq \gamma_c(L^{(S)}(G))$. Conversely, if $u = u(y_1G', \ldots, y_nG')$ is a Lie commutator of length c in the $y_1G', \ldots, y_nG'$ then (since $\gamma_{c+1}(L^{(S)}(G)) = \{0\}$) it is equal to the group commutator in the $y_1G', \ldots, y_nG'$ obtained from u by replacing the operation on ring multiplication by the operation of commutation in the group H. Since every element of $\gamma_c(L^{(S)}(G))$ is a linear combination with integer coefficients of Lie commutators of length c in the $y_1G', \ldots, y_nG'$, we have the inverse inclusion $\gamma_c(L^{(S)}(G)) \subseteq \gamma_c(H)$. Therefore, we get $\gamma_c(H) = \gamma_c(L^{(S)}(G)) = \gamma_c(G)$. Write $G^{(c-1)} = G/\tau_c(G)$. Thus $G^{(c-1)}$ is a relatively free nilpotent torsion-free group of rank n and class $c-1$. It is easy to verify that $\tau_{i+1}(G^{(c-1)}) = \tau_{i+1}(G)/\tau_i(G)$ for $i = 1, \ldots, c-1$. Since $(G^{(c-1)})' = G'/\tau_c(G)$, we get $G^{(c-1)}/(G^{(c-1)})'$ is a free abelian group of rank n. Moreover $G^{(c-1)}$ is generated by the set $\{\overline{y}_1, \ldots, \overline{y}_n\}$, where $\overline{y}_i = y_i\tau_c(G)$, $i = 1, \ldots, n$ and $\tau_2(G^{(c-1)}) = G'/\tau_c(G) = (G^{(c-1)})'$. Further $L^{(S)}(G^{(c-1)})$ is free abelian, $L_{\mathbb{Q}}(G^{(c-1)}) = \mathbb{Q} \otimes_{\mathbb{Z}} L^{(S)}(G^{(c-1)})$, and $L^{(S)}(G^{(c-1)})$ is regarded as a subset of $L_{\mathbb{Q}}(G^{(c-1)})$. By Lemma 3.2, $L^{(S)}(G^{(c-1)})$ is generated as a Lie ring by the set $\{\overline{y}_1(G^{(c-1)})', \ldots, \overline{y}_n(G^{(c-1)})'\}$. It is easy to check that $\{\overline{y}_1(G^{(c-1)})', \ldots, \overline{y}_n(G^{(c-1)})'\}$ is a $\mathbb{Z}$-basis for $G^{(c-1)}/(G^{(c-1)})'$ and so, by Lemma 3.3, $\{\overline{y}_1(G^{(c-1)})' + L_{\mathbb{Q}}(G^{(c-1)})', \ldots, \overline{y}_n(G^{(c-1)})' + L_{\mathbb{Q}}(G^{(c-1)})'\}$ is a $\mathbb{Q}$-basis for $L_{\mathbb{Q}}(G^{(c-1)})/L_{\mathbb{Q}}(G^{(c-1)})'$. As before, we give on $L_{\mathbb{Q}}(G^{(c-1)})$ the structure of a group by means of the BCH formula, denoted by $R^{(c-1)}$. Let $H^{(c-1)}$ be the subgroup of $R^{(c-1)}$ generated by the set $\{\overline{y}_1(G^{(c-1)})', \ldots, \overline{y}_n(G^{(c-1)})'\}$. By Lemma 2.1, $H^{(c-1)}$ is a torsion-free finitely generated nilpotent group and $\tau_2(H^{(c-1)}) = (H^{(c-1)})'$. Using similar arguments as before, $\gamma_{c-1}(H^{(c-1)}) = \gamma_{c-1}(L^{(S)}(G^{(c-1)})$. In the next few lines, we write $b_i = y_iG'$ and $b_i^{(c-1)} = \overline{y}_i(G^{(c-1)})'$, $i = 1, \ldots, n$. As the group operation $\circ$ can be expressed in terms of the Lie algebra operation, the natural Lie algebra epimorphism α_L from $L_{\mathbb{Q}}(G)$ onto $L_{\mathbb{Q}}(G^{(c-1)})$ induces a group epimorphism α_H from H onto $H^{(c-1)}$ such that

$\alpha_H(y_iG') = \alpha_L(y_iG') = \overline{y}_i(G^{(c-1)})'$, $i = 1, \ldots, n$. We claim that the kernel of α_H, $\ker\alpha_H$, is equal to $\gamma_c(H)$. Since $\gamma_c(H) = \gamma_c(L^{(S)}(G)) = \gamma_c(G)$ and $\gamma_c(G) \subseteq \ker\alpha_H$, it is enough to show that $\ker\alpha_H \subseteq \gamma_c(G)$. Let $v \in \ker\alpha_H$ and let μ be the smallest natural integer such that $v \in \gamma_\mu(H) \setminus \gamma_{\mu+1}(H)$. Write

$$v = (\prod_{\text{finite}} (b_{i_1}, \ldots, b_{i_\mu})^{a_{(i_1,\ldots,i_\mu)}})v'(b_1, \ldots, b_n),$$

where $a_{(i_1,\ldots,i_\mu)} \in \mathbb{Z}$ and $v'(b_1, \ldots, b_n) \in \gamma_{\mu+1}(H)$. Since $v \in \ker\alpha_H$, we obtain

$$(\prod_{\text{finite}} (b_{i_1}^{(c-1)}, \ldots, b_{i_\mu}^{(c-1)})^{a_{(i_1,\ldots,i_\mu)}})v'(b_1^{(c-1)}, \ldots, b_n^{(c-1)}) = 0.$$

By the BCH formula,

$$\sum a_{(i_1,\ldots,i_\mu)}[b_{i_1}^{(c-1)}, \ldots, b_{i_\mu}^{(c-1)}] \in \gamma_{\mu+1}(L_{\mathbb{Q}}(G^{(c-1)})).$$

Since $L_{\mathbb{Q}}(G^{(c-1)}) = \oplus_{1\leq i\leq c-1}(\mathbb{Q} \otimes L_i^{(S)}(G^{(c-1)}))$ and, for $t = 1, \ldots, c-1$, $\gamma_t(L_{\mathbb{Q}}(G^{(c-1)}) = \oplus_{i\geq t}(\mathbb{Q} \otimes L_i^{(S)}(G^{(c-1)}))$, we have

$$\sum a_{(i_1,\ldots,i_\mu)}[b_{i_1}^{(c-1)}, \ldots, b_{i_\mu}^{(c-1)}] = 0$$

and so,

$$\sum a_{(i_1,\ldots,i_\mu)}[b_{i_1}, \ldots, b_{i_\mu}] \in \ker\alpha_L,$$

where $\ker\alpha_L$ denotes the kernel of α_L. Suppose that $\mu \neq c$. Since $\ker\alpha_L = \mathbb{Q} \otimes_{\mathbb{Z}} \gamma_c(G)$ and $L_{\mathbb{Q}}(G)$ is graded, we obtain

$$\sum a_{(i_1,\ldots,i_\mu)}[b_{i_1}, \ldots, b_{i_\mu}] = 0 \ \ (\text{in} \ \ L_{\mathbb{Q}}(G)).$$

Thus

$$\prod_{\text{finite}} (y_{i_1}, \ldots, y_{i_\mu})^{a_{(i_1,\ldots,i_\mu)}} \in \gamma_{\mu+1}(G).$$

Recall from the proof of Theorem 3 that η is the Lie algebra isomorphism from $\mathcal{L}_{\mathbb{Q}}(G)$ into $L_{\mathbb{Q}}(G)$ satisfying the conditions $\eta(\log y_i) = y_iG'$ for $i = 1, \ldots, n$. Let ξ_η be the mapping from the group $\exp \mathcal{L}_{\mathbb{Q}}(G)$ to the group R defined by $\xi_\eta(\exp u) = \eta(u)$ for all $u \in \mathcal{L}_{\mathbb{Q}}(G)$. Since η is a Lie algebra homomorphism and the group operation $\circ$ is expressed in terms of Lie commutators, we have $\xi_\eta((\exp u)(\exp v)) = \eta(u) \circ \eta(v)$ for all $u, v \in \mathcal{L}_{\mathbb{Q}}(G)$. It is easily

verified that ξ_η is a group isomorphism. But $\xi_\eta(y_i) = \eta(\log y_i) = b_i$ for $i = 1, \ldots, n$. Therefore $\xi_\eta(G) = H$ and so, for $t = 1, \ldots, c$, $\xi_\eta(\gamma_{t+1}(G)) = \gamma_{t+1}(H)$. Hence

$$\prod_{\text{finite}} (b_{i_1}, \ldots, b_{i_\mu})^{a(i_1,\ldots,i_\mu)} \in \gamma_{\mu+1}(H),$$

which is a contradiction. Therefore, $\mu = c$ and $v \in \gamma_c(H) = \gamma_c(G)$. Hence $\ker\alpha_H \subseteq \gamma_c(G)$ and $\ker\alpha_H = \gamma_c(G) = \gamma_c(H)$. Thus

$$\gamma_{c-1}(H^{(c-1)}) = \gamma_{c-1}(H)/\gamma_c(H) \cong \gamma_{c-1}(L^{(S)}(G))/\gamma_c(L^{(S)}(G)).$$

Eventually we see that the isomorphism (16) holds for every $i \leq c$. Since

$$\gamma_i(L^{(S)}(G))/\gamma_{i+1}(L^{(S)}(G)) \cong \gamma_i(G)\tau_{i+1}(G)/\tau_{i+1}(G)$$

for $i = 1, \ldots, c$, we obtain from (16) that H is Magnus. Since $G \cong H$ by means of ξ_η, we have the required result.

(II) Since $H \cong F_{n,c}/\ker\tau$ and $\ker\tau$ is fully invariant (by Proposition 3.3 (iii)), we obtain H is relatively free of finite rank. Furthermore, since H is torsion-free nilpotent, we obtain from (I) that H is Magnus. By Lemma 3.10, we obtain $\mathcal{L}_{\mathbb{Q}}(\ker\tau) \subseteq \mathcal{L}_{\mathbb{Q}}(F_{n,c})'$. By Lemma 2.1 and (I), $L \cong \mathcal{L}_{\mathbb{Q}}(H) \cong L_{\mathbb{Q}}(H)$ as Lie algebras. $\square$

Remark 4.1 Let L be a relatively free nilpotent Lie algebra over $\mathbb{Q}$ of finite rank n, with $n \geq 2$. Let $\{h_1, \ldots, h_n\}$ be a free generating set of L. Then $\{h_1 + L', \ldots, h_n + L'\}$ is a $\mathbb{Q}$-basis for L. Let $y_1, \ldots, y_n$ be elements of L such that the set $\{y_1 + L', \ldots, y_n + L'\}$ is a $\mathbb{Q}$-basis of L. For each j, with $j = 1, \ldots, n$,

$$h_j = \sum_{i=1}^{n} \alpha_{ij} y_i + v_i,$$

where $v_i \in L'$, $i = 1, \ldots, n$, and $\alpha_{ij} \in \mathbb{Q}$. It is easily verified that L is generated by the set $\{y_1, \ldots, y_n\}$. Let φ be the map from L into L satisfying the conditions $\varphi(h_j) = y_j$, $j = 1, \ldots, n$. Since L is relatively free on $\{h_1, \ldots, h_n\}$ and L is generated by the set $\{y_1, \ldots, y_n\}$, φ extends uniquely to a Lie algebra epimorphism of L. Since φ induces a group automorphism on L/L' and L is nilpotent, it is easily checked that φ is an automorphism of L. Thus the set $\{y_1, \ldots, y_n\}$ is a free generating set of L. Consider L as a group, denoted R, by means of the BCH formula. Let H_1 and H_2 be the subgroups of R generated by the sets $\{h_1, \ldots, h_n\}$ and $\{y_1, \ldots, y_n\}$, respectively. By the proof of Theorem A (II) and since both H_1 and H_2 have

rank n, we get $H_1 \cong H_2$. Hence for a relatively free nilpotent Lie algebra of finite rank n over $\mathbb{Q}$, we associate (via BCH formula) a unique (up to isomorphism) relatively free Magnus nilpotent group of rank n.

Proof of Corollary 1.1. Let G be a torsion-free finitely generated nilpotent group of class c, and let K be a field of characteristic zero. By Lemma 3.7, we obtain $\mathrm{grad}^{(\ell)}(\mathcal{L}_K(G)) \cong L_K(G)$ as Lie algebras in a natural way. Write $R_K(G) = \exp \mathcal{L}_K(G)$, and give on $\mathcal{L}_K(G)$ the structure of a group by means of the BCH formula. Then $(\mathcal{L}_K(G), \circ)$ is isomorphic to $R_K(G)$ by a group isomorphism sending u to $\exp u$ for all $u \in \mathcal{L}_K(G)$. Thus $\exp \gamma_t(\mathcal{L}_K(G)) = \gamma_t(R_K(G))$ for $t = 1, \ldots, c$. Form the direct sum of the abelian groups

$$\mathrm{grad}^{(g)}(R_K(G)) = \oplus_{t=1}^{c} \gamma_t(R_K(G))/\gamma_{t+1}(R_K(G))$$

and give it the structure of a Lie algebra by defining a Lie multiplication

$$[u\gamma_{i+1}(R_K(G)), v\gamma_{j+1}(R_K(G))] = (u, v)\gamma_{i+j+1}(R_K(G))$$

for $u \in \gamma_i(R_K(G))$, $v \in \gamma_j(R_K(G))$, $i, j \in \{1, \ldots, c\}$. Extend this multiplication to $\mathrm{grad}^{(g)}(R_K(G))$ by linearity. Similarly, we form the direct sum of the abelian groups

$$\mathrm{grad}^{(g)}(\mathcal{L}_K(G)) = \oplus_{t=1}^{c} \gamma_t(\mathcal{L}_K(G))/\gamma_{t+1}(\mathcal{L}_K(G))$$

and give it the structure of a Lie algebra. Since $(\mathcal{L}_K(G), \circ) \cong R_K(G)$ as groups, we have

$$\mathrm{grad}^{(g)}(R_K(G)) \cong \mathrm{grad}^{(g)}(\mathcal{L}_K(G))$$

as Lie algebras in a natural way. Let $\{a_1, \ldots, a_{f(c+1)}\}$ be a canonical basis for G. Then the set $\{\log a_1, \ldots, \log a_{f(c+1)}\}$ is a K-basis of $\mathcal{L}_K(G)$. By Lemma 3.5 and the proof of Lemma 3.6, $\gamma_t(\mathcal{L}_K(G))/\gamma_{t+1}(\mathcal{L}_{t+1}(G))$ has a K-basis the set $\{\log a_{f(c+1)} + \gamma_{t+1}(\mathcal{L}_K(G)), \ldots, \log a_{f(c+1)} + \gamma_{t+1}(\mathcal{L}_K(G))\}$. Using the BCH formula,

$$(\lambda_1 \log a_{f(t)+1} + \cdots + \lambda_{n_t} \log a_{f(t+1)}) + \gamma_{t+1}(\mathcal{L}_K(G)) =$$

$$(\lambda_1 \log a_{f(t)+1} \circ \cdots \circ \lambda_{n_t} \log a_{f(t+1)}) \circ \gamma_{t+1}(\mathcal{L}_K(G))$$

(as sets). Therefore every element of $(\mathcal{L}_K(G), \circ)$ is written uniquely as $\lambda_1 \log a_1 \circ \cdots \circ \lambda_{f(c+1)} \log a_{f(c+1)}$, where $\lambda_1, \ldots, \lambda_{f(c+1)} \in K$. Hence

$$\mathrm{grad}^{(g)}(\mathcal{L}_K(G)) = \mathrm{grad}^{(\ell)}(\mathcal{L}_K(G))$$

as Lie algebras. Since $(\mathcal{L}_K(G)), \circ) \cong R_K(G)$, we have every element of $R_K(G)$ is written uniquely as $\exp(\lambda_1 \log a_1) \cdots \exp(\lambda_{f(c+1)} \log a_{f(c+1)})$ with $\lambda_1, \ldots, \lambda_{f(c+1)} \in K$ and so,

$$\mathrm{grad}^{(g)}(R_K(G)) \cong \mathrm{grad}^{(\ell)}(\mathcal{L}_K(G))$$

as Lie algebras in a natural way. Since $\mathrm{grad}^{(\ell)}(\mathcal{L}_K(G)) \cong L_K(G)$ (as Lie algebras), we have $\mathrm{grad}^{(g)}(R_K(G)) \cong L_K(G)$ as Lie algebras in a natural way.

(II) Suppose that G is relatively free. By Theorem 3, we obtain $\mathcal{L}_K(G) \cong L_K(G)$ as Lie algebras and so, by (I), $\mathcal{L}_K(G) \cong \mathrm{grad}^{(g)}(\exp \mathcal{L}_K(G))$.

Remark 4.2 It is easy to verify that

$$K \otimes_{\mathbb{Q}} \mathrm{grad}^{(g)}(\exp \mathcal{L}_{\mathbb{Q}}(G)) \cong \mathrm{grad}^{(g)}(\exp \mathcal{L}_K(G)).$$

as Lie algebras, where K is a field of characteristic zero. For $K = \mathbb{R}$, $\exp \mathcal{L}_{\mathbb{R}}(G)$ is a real simply connected Lie group whose rational Lie algebra is $\mathcal{L}_{\mathbb{Q}}(G)$. Moreover, if $\{a_1, \ldots, a_{f(c+1)}\}$ is a canonical basis of G, then every element of $\exp \mathcal{L}_{\mathbb{R}}(G)$ is written uniquely as

$$\exp(\lambda_1 \log a_1) \cdots \exp(\lambda_{f(c+1)} \log a_{f(c+1)})$$

with $\lambda_1, \ldots, \lambda_{f(c+1)} \in \mathbb{R}$. Next, we recall a standard procedure of a construction of a Lie group from a finite-dimensional nilpotent Lie algebra L over $\mathbb{Q}$. Let $b_1, \ldots, b_n \in L$ such that the set $\{b_1 + L', \ldots, b_n + L'\}$ is a $\mathbb{Q}$-basis of L/L'. Give on L the structure of a group via the BCH formula. Let H be the subgroup of $(L, \circ)$ generated by the set $\{b_1, \ldots, b_n\}$. Then $(L, \circ)$ is a Mal'cev completion of H, and $(\mathbb{R} \otimes_{\mathbb{Q}} L, \circ)$ is a real simply connected Lie group containing H as a discrete subgroup with rational Lie algebra L.

Proof of Corollary 1.2. Let G_1, G_2 be torsion-free finitely generated nilpotent groups which are quasi-isometric. By Remark 4.2, $\exp \mathcal{L}_{\mathbb{R}}(G_j)$, with $j = 1, 2$ is a real simply connected Lie group whose rational Lie algebra is $\mathcal{L}_{\mathbb{Q}}(G_j)$. It follows from a result of Pansu [17, Theorem 3] that

$$\mathrm{grad}^{(g)}(\exp \mathcal{L}_{\mathbb{R}}(G_1)) \cong \mathrm{grad}^{(g)}(\exp \mathcal{L}_{\mathbb{R}}(G_2))$$

as Lie algebras. Suppose that G_j $(j = 1, 2)$ is a relatively free group of finite rank. By Theorem A (I), G_j is a Magnus group, with $j = 1, 2$. By Proposition 3.2 and Theorem 3,

$\mathcal{L}_{\mathbb{R}}(G_j)$ is relatively free and $\mathcal{L}_{\mathbb{R}}(G_j) \cong L_{\mathbb{R}}(G_j)$ as Lie algebras for $j = 1, 2$. By Corollary 1.1, $\mathcal{L}_{\mathbb{R}}(G_2) \cong \mathcal{L}_{\mathbb{R}}(G_1)$ as Lie algebras. Hence both G_2 and G_1 have the same finite rank n and nilpotency class c, and $\gamma_i(G_2)/\gamma_{i+1}(G_2) \cong \gamma_i(G_1)/\gamma_{i+1}(G_1)$, $i = 1, \ldots, c$. Let $F_{n,c}$ be the free nilpotent group of rank n and class c; freely generated by the set $\{x_1, \ldots, x_n\}$. For $j = 1, 2$, let $\{y_{1j}, \ldots, y_{nj}\}$ be a free generating set for G_j. Furthermore, we write π_j for the natural group epimorphism from $F_{n,c}$ onto G_j such that $\pi_j(x_i) = y_{ij}$, $i = 1, \ldots, n$. Thus $G_j \cong F_{n,c}/\mathrm{ker}\pi_j$, with $j = 1, 2$. By Lemma 3.4 and the proof of Theorem 3, the set $\{\log y_{1j}, \ldots, \log y_{nj}\}$ is a free generating set for $\mathcal{L}_{\mathbb{Q}}(G_j)$. We claim $\mathcal{L}_{\mathbb{Q}}(G_2) \cong \mathcal{L}_{\mathbb{Q}}(G_1)$ as Lie algebras. To get a contradiction we assume that $\mathcal{L}_{\mathbb{Q}}(G_2) \ncong \mathcal{L}_{\mathbb{Q}}(G_1)$, and let w be a word (for Lie algebras over $\mathbb{Q}$) such that $w(\log y_{11}, \ldots, \log y_{n1}) = 0$ and $w(\log y_{12}, \ldots, \log y_{n2}) \neq 0$. Since $\mathcal{L}_{\mathbb{R}}(G_j) = \mathbb{R} \otimes_{\mathbb{Q}} \mathcal{L}_{\mathbb{Q}}(G_j)$, $j = 1, 2$, and $\mathcal{L}_{\mathbb{R}}(G_1) \cong \mathcal{L}_{\mathbb{R}}(G_2)$ (as Lie algebras), we obtain w is an identity in $\mathcal{L}_{\mathbb{R}}(G_2)$ which is a contradiction. Since both $\mathcal{L}_{\mathbb{Q}}(G_1)$ and $\mathcal{L}_{\mathbb{Q}}(G_2)$ are relatively free of finite rank n in the same variety, we get $\mathcal{L}_{\mathbb{Q}}(G_1) \cong \mathcal{L}_{\mathbb{Q}}(G_2)$. Since, G_1 and G_2 are relatively free, in order to prove that G_1 is isomorphic to G_2, it is enough to show that $\mathrm{ker}\pi_1 \subseteq \mathrm{ker}\pi_2$. Let $\widetilde{w} \in \mathrm{ker}\pi_1 \subseteq F'_{n,c}$. By Lemma 3.9 (for $M = G_1$), the natural epimorphism π_1 gives rise a Lie algebra homomorphism ψ_{π_1} from $\mathcal{L}_{\mathbb{Q}}(F_{n,c})$ into $\mathcal{L}_{\mathbb{Q}}(G_1)$ such that $\psi_{\pi_1}(\log u) = \log \pi_1(u)$ for all $u \in \mathrm{ker}\pi_1$. Thus $\psi_{\pi_1}(\log \widetilde{w}) = 0$ and so, $\log \widetilde{w} \in \mathcal{L}_{\mathbb{Q}}(\mathrm{ker}\pi_1)$. Thus

$$\log \widetilde{w} = \widetilde{w}_2 + \cdots + \widetilde{w}_c,$$

where $\widetilde{w}_t \in \mathcal{L}_{\mathbb{Q}}(\mathrm{ker}\pi_1) \cap \mathcal{L}_t(F_{n,c})$ (by Lemma 3.11) for $t = 2, \ldots, c$. By Proposition 3.1, $\widetilde{w}_2, \ldots, \widetilde{w}_c$ are identities in $\mathcal{L}_{\mathbb{Q}}(G_1)$. Since $\mathcal{L}_{\mathbb{Q}}(G_1) \cong \mathcal{L}_{\mathbb{Q}}(G_2)$, we have $\widetilde{w}_2, \ldots, \widetilde{w}_c$ are identities for $\mathcal{L}_{\mathbb{Q}}(G_2)$ as well. Therefore $\widetilde{w}_t \in \mathcal{L}_{\mathbb{Q}}(\mathrm{ker}\pi_2) \cap \mathcal{L}_t(F_{n,c})$ for $t = 2, \ldots, c$. Hence $\log \widetilde{w} \in \mathcal{L}_{\mathbb{Q}}(\mathrm{ker}\pi_2)$. Thus $\psi_{\pi_2}(\log \widetilde{w}) = 0$ and so, $\widetilde{w} \in \mathrm{ker}\pi_2$. Therefore, there exists a group epimorphism from G_1 onto G_2. Since $\gamma_i(G_1)/\gamma_{i+1}(G_1) \cong \gamma_i(G_2)/\gamma_{i+1}(G_2)$, $i = 1, \ldots, c$, we obtain $G_1 \cong G_2$. $\square$

4.2 Proof of Theorem B.

(I) It has been proved in Theorems 4 and 5.

(II) Let L be a relatively free Lie algebra over $\mathbb{Q}$ of rank n, with $n \geq 2$. If L is nilpotent, then our claim follows from Theorem A (II). Thus we assume that L is not nilpotent. Let $\mathfrak{V}$ be a variety of Lie algebras such that L is relatively free of rank n.

Therefore $L \cong L_n/\mathfrak{V}(L_n)$, where L_n is the absolutely free Lie algebra freely generated by the set $\{\ell_1, \ldots, \ell_n\}$. Without loss of generality, we may write $L = L_n/\mathfrak{V}(L_n)$. Let $y_i = \ell_i + \mathfrak{V}(L_n)$, $i = 1, \ldots, n$. Thus the set $\{y_1, \ldots, y_n\}$ is a free generating set of L. For a positive integer m, let L_n^m be the subspace of L_n spanned by all Lie commutators of total degree m in $\ell_1, \ldots, \ell_n$. Since $\mathbb{Q}$ is infinite, $L_n \cap \mathfrak{V}(L_n) = \oplus_{m\geq 1}(L_n^m \cap \mathfrak{V}(L_n))$. Thus we may write L as a sum of homogeneous components, $L = \oplus_{m\geq 1} L^m$, where $L^m \cong L_n^m/(L_n^m \cap \mathfrak{V}(L_n))$ and L^m is the subspace of L spanned by all Lie commutators of total degree m in $y_1, \ldots, y_n$. Each element u of L may be uniquely written in the form $u = \sum_{m\geq 1} u_m$ with $u_m \in L^m$ for all m and $u_m = 0$ for all but finitely many m. Furthermore, for $c \geq 1$, $\gamma_c(L) = \oplus_{m\geq c} L^m$. We write $L(c) = L/\gamma_{c+1}(L)$, and $y_{i,c} = y_i + \gamma_{c+1}(L)$, $i = 1, \ldots, n$. Since L is relatively free, then $L(c)$ is a relatively free nilpotent Lie algebra of rank n and class c with a free generating set $\{y_{1,c}, \ldots, y_{n,c}\}$. Notice that $\{y_{1,c} + L(c)', \ldots, y_{n,c} + L(c)'\}$ is a basis of $L(c)/L(c)'$. Give on $L(c)$ the structure of a group, denoted by R_c, by means of the BCH formula, and let Y_c be the subgroup of R_c generated by the set $\{y_{1,c}, \ldots, y_{n,c}\}$. By Theorem A (II), Y_c is a finitely generated Magnus nilpotent group of class c. By the BCH formula, $\gamma_c(L(c))$ is spanned by all group commutators $(y_{i_1,c}, \ldots, y_{i_c,c})$ with $i_1, \ldots, i_c \in \{1, \ldots, n\}$. For positive integers c and d, with $c \leq d$, let $\rho_{c,d}$ be the natural Lie algebra epimorphism from $L(d)$ onto $L(c)$ sending $y_{i,d}$ to $y_{i,c}$ for $i = 1, \ldots, n$. As the group operation in Y_c (for all c) can be expressed in terms of the Lie algebra operations, $\rho_{c,d}$ induces a group homomorphism, say $\widetilde{\rho}_{c,d}$, from Y_d onto Y_c such that $\widetilde{\rho}_{c,d}$ sends $y_{i,d}$ to $y_{i,c}$ for $i = 1, \ldots, n$. It is clearly enough that $Y_d/\gamma_d(Y_d) \cong Y_{d-1}$. Since $\gamma_d(L(d))$ is spanned by all group commutators $(y_{i_1,d}, \ldots, y_{i_d,d})$, with $i_1, \ldots, i_d \in \{1, \ldots, n\}$, and $\ker\rho_{(d-1),d} = \gamma_d(L(d))$, we obtain $\gamma_d(Y_d) \subseteq \ker\widetilde{\rho}_{(d-1),d}$. Since $Y_d/\ker\widetilde{\rho}_{(d-1),d} \cong Y_{d-1}$ and each Y_d is torsion-free finitely generated nilpotent, we get the kernel of $\widetilde{\rho}_{(d-1),d}$ is equal to $\gamma_d(Y_d)$ for all $d \geq 2$.

Let $\widehat{L} = \varprojlim L(c)$ be the completion of L with respect to the lower central series. ($\varprojlim L(c)$ may be identified with the complete (unrestricted) direct sum $\widehat{\oplus}_{m\geq 1} L^m$, and it has a natural Lie algebra structure.) Moreover, L is naturally contained in $\widehat{L}$. Give on $\widehat{L}$ the structure of a group, denoted by $\widehat{R}$, via the BCH formula. That is, $\widehat{R} = (\widehat{L}, \circ) = \varprojlim R(c)$. Let H be the subgroup of $\widehat{R}$ generated by the set $\{y_1, \ldots, y_n\}$. Notice that, for $i = 1, \ldots, n$,

$$y_i = (y_{i,1}, y_{i,2}, \ldots),$$

and, for $i, j \in \{1, \ldots, n\}$,

$$y_i \circ y_j = (y_{i,1} \circ y_{j,1}, y_{i,2} \circ y_{j,2}, \cdots).$$

Moreover, $H \subseteq \prod_{c \geq 1} Y_c$, and it is easy to verify that $H = \varprojlim(Y_c, \widetilde{\rho}_{c,d})$.

Let F_n be a free group of rank n, with $n \geq 2$, freely generated by the set $\{f_1, \ldots, f_n\}$. We write $F_{n,c}$ for the free nilpotent group of rank n and class c; freely generated by the set $\{x_1, \ldots, x_n\}$, with $x_i = f_i \gamma_{c+1}(F_n)$, $i = 1, \ldots, n$. Let α_c be the natural epimorphism from $F_{n,c}$ onto Y_c sending x_i to $y_{i,c}$, $i = 1, \ldots, n$. Then $F_{n,c}/\ker\alpha_c \cong Y_c$ via an isomorphism $\widetilde{\alpha}_c$ induced by α_c, and $\ker\alpha_c$ is a fully invariant subgroup of $F_{n,c}$. Let π_c be the natural epimorphism from F_n onto $F_{n,c}$ sending f to $f\gamma_{c+1}(F_n)$ for all $f \in F_n$, and let $\delta_c = \alpha_c \pi_c$. Thus $F_n/\ker\delta_c \cong Y_c$ by an isomorphism $\widetilde{\delta}_c$ induced by δ_c. The group $\ker\delta_c$ is a fully invariant subgroup of F_n since Y_n is a relatively free group. So we have

$$\begin{tikzcd}
F_n \arrow[r, "\pi_c"] \arrow[dr, "\delta_c"'] & F_{n,c} \arrow[d, "\alpha_c"] & \\
& Y_c & Y_d \arrow[l, "\widetilde{\rho}_{c,d}"']
\end{tikzcd}$$

Denote $h_{i,c} = f_i \ker\delta_c$ with $i = 1, \ldots, n$. Notice that $\widetilde{\delta}_c(h_{i,c}) = \delta_c(f_i) = y_{i,c}$ for $i = 1, \ldots, n$. Let $c \leq d$, and let $w(f_1, \ldots, f_n) \in \ker\delta_d$. Then $\delta_d(w(f_1, \ldots, f_n)) = w(y_{1,d}, \ldots, y_{n,d}) = 1_{Y_d}$. Since $\widetilde{\rho}_{c,d}$ is a group homomorphism, we have $w(y_{1,c}, \ldots, y_{n,c}) = 1_{Y_c}$ and so, $\ker\delta_d \subseteq \ker\delta_c$. Set $\psi_{c,d} = (\widetilde{\delta}_c)^{-1} \widetilde{\rho}_{c,d} \widetilde{\delta}_d$.

$$\begin{tikzcd}
F_n/\ker\delta_d \arrow[r, "\widetilde{\delta}_d"] \arrow[d, "\psi_{c,d}"'] & Y_d \arrow[d, "\widetilde{\rho}_{c,d}"] \\
F_n/\ker\delta_c \arrow[r, "\widetilde{\delta}_c"] & Y_c
\end{tikzcd}$$

It is clearly enough that $\psi_{c,d}(x\ker\delta_d) = x\ker\delta_c$ for all $x \in F_n$. In particular, $\psi_{c,d}(h_{i,d}) = h_{i,c}$ for $i = 1, \ldots, n$.

For the rest of the proof, we identify Y_c with $F_n/\ker\delta_c$ under the group isomorphism $\widetilde{\delta}_c$. In the light of this identification, $y_{i,c} = h_{i,c}$ and $y_i = (h_{1,c}, h_{2,c}, \ldots)$ with $i = 1, \ldots, n$. Moreover $H = \varprojlim(F_n/\ker\delta_c, \psi_{c,d})$. Throughout the proof we use both Y_c and $F_{n,c}/\ker\delta_c$ without making any distinction. We claim that H is a relatively free residually

torsion-free nilpotent group of rank n. First we shall show that H is relatively free of rank n. Let ϑ_n be the natural homomorphism from F_n into H sending $w = w(f_1, \ldots, f_n)$ to $(w\ker\delta_1, w\ker\delta_2, \ldots)$. Since H is generated by the set $\{y_1, \ldots, y_n\}$, we have $F_n/(\cap\ker\delta_c) \cong H$ by the isomorphism $\widetilde{\vartheta}_n$ induced by ϑ_n. Set $M = \cap\ker\delta_c$. Since M is a fully invariant subgroup of F_n, we obtain H is relatively free of rank n.

Let $u = u(y_1, \ldots, y_n) \in H$. Since H is generated by the set $\{y_1, \ldots, y_n\}$, u is written as $u = y_{i_1}^{a_1} \cdots y_{i_\kappa}^{a_\kappa}$ with $a_1, \ldots, a_\kappa \in \mathbb{Z}$ and $i_1, \ldots, i_\kappa \in \{1, \ldots, n\}$. Then

$$u = u(y_1, \ldots, y_n) = (u(h_{1,1}, \ldots, h_{n,1}), \ldots, u(h_{1,c}, \ldots, h_{n,c}), \ldots).$$

Notice that $u(h_{1,c}, \ldots, h_{n,c}) \in Y_c$ for all c and, for $c \leq d$, $\psi_{c,d}(u(h_{1,d}, \ldots, h_{n,d})) = u(h_{1,c}, \ldots, h_{n,c})$. Suppose that $y_1^{a_1} \cdots y_n^{a_n} \in H'$ for some $a_1, \ldots, a_n \in \mathbb{Z}$. Then $h_{1,1}^{a_1} \cdots h_{n,1}^{a_n} \in Y_1'$. Since Y_1/Y_1' is free abelian, we obtain H/H' is a free abelian group of rank n. Suppose that $\gamma_s(H) = \gamma_{s+1}(H)$ for some s. Then $(y_{i_1}, \ldots, y_{i_s}) \in \gamma_{s+1}(H)$ for all $i_1, \ldots, i_s \in \{1, \ldots, n\}$. Then $(h_{i_1,s}, \ldots, h_{i_s,s}) = 1_{Y_s}$ for all $i_1, \ldots, i_s \in \{1, \ldots, n\}$. Since Y_s is freely generated by the set $\{h_{1,s}, \ldots, h_{n,s}\}$, we obtain Y_s has nilpotency class $s-1$ which is a contradiction. Therefore $\gamma_s(H) \neq \gamma_{s+1}(H)$ for all s and so, there are no repetitions of terms of the lower central series of H.

Our next step is to prove that H is residually torsion-free nilpotent. For a positive integer c, let ζ_c be the natural epimorphism from F_n/M onto Y_c. (That is, $\zeta_c(wM) = w\ker\delta_c$ for all $w \in F_n$.) For $c \leq d$, it is easily verified that $\psi_{c,d}\zeta_d(fM) = \zeta_c(fM)$ for all $f \in F_n$. Write $\widetilde{\zeta}_c = \zeta_c(\widetilde{\vartheta}_n)^{-1}$. Thus $\widetilde{\zeta}_c(y_i) = h_{i,c}$ for $i = 1, \ldots, n$. Let $N_c = H/\gamma_{c+1}(H)$. The group N_c has nilpotency class c since $\gamma_s(H) \neq \gamma_{s+1}(H)$ for all s. Let β_c be the natural epimorphism from $F_{n,c}$ onto N_c such that $\beta_c(x_i) = \overline{y}_i$, where $\overline{y}_i = y_i\gamma_{c+1}(H)$ $i = 1, \ldots, n$. Since $F_{n,c}/F_{n,c}' \cong Y_c/Y_c' \cong N_c/N_c'$, we obtain both $\ker\alpha_c$ and $\ker\beta_c$ are subgroups of $F_{n,c}'$.

$$\begin{tikzcd}
F_{n,c} \arrow[r, "\alpha_c"] \arrow[d, "\beta_c"'] & Y_c \\
N_c \arrow[ur, dotted, "\xi_c"'] &
\end{tikzcd}$$

We claim that $\ker\beta_c \subseteq \ker\alpha_c$. Let $v = v(x_1, \ldots, x_n) \in \ker\beta_c$. Then $\beta_c(v) = v(\overline{y}_1, \ldots, \overline{y}_n) = 1_{N_c}$ or, equivalently, $v(y_1, \ldots, y_n) \in \gamma_{c+1}(H)$. Thus $v(h_{1,q}, \ldots, h_{n,q}) \in \gamma_{c+1}(Y_q)$ for all q. Hence $v(h_{1,c}, \ldots, h_{n,c}) = 1_{Y_c}$ and so, $v = v(x_1, \ldots, x_n) \in \ker\alpha_c$. Therefore $\ker\beta_c \subseteq \ker\alpha_c$.

Let γ_c be the natural epimorphism from $F_{n,c}/\mathrm{ker}\beta_c$ onto $F_{n,c}/\mathrm{ker}\alpha_c$. Moreover, we write $\xi_c = \widetilde{\alpha}_c\gamma_c\widetilde{\beta}_c^{-1}$, where $\widetilde{\alpha}_c$ and $\widetilde{\beta}_c$ are the isomorphisms induced by α_c and β_c respectively.

$$\begin{array}{ccc} F_{n,c}/\mathrm{ker}\beta_c & \xrightarrow{\widetilde{\beta}_c} & N_c \\ \Big\downarrow{\scriptstyle \gamma_c} & & \Big\downarrow{\scriptstyle \xi_c} \\ F_{n,c}/\mathrm{ker}\alpha_c & \xrightarrow{\widetilde{\alpha}_c} & Y_c \end{array}$$

It is easily verified that $\xi_c(v(y_1,\ldots,y_n)\gamma_{c+1}(H)) = v(y_{1,c},\ldots,y_{n,c})$ for $v(y_1,\ldots,y_n) \in H$. Let $u = u(y_1,\ldots,y_n) \in \tau_{c+1}(H)$. Then there exists $m \in \mathbb{N}$ and $v \in \gamma_{c+1}(H)$ such that $u^m = v$. By applying ξ_c, we have

$$\begin{aligned} 1_{Y_c} &= \xi_c(u^m\gamma_{c+1}(H)) \\ &= \xi_c(u\gamma_{c+1}(H))^m \\ &= u(y_{1,c},\ldots,y_{n,c})^m. \end{aligned}$$

Since Y_c is torsion-free, we get $u(y_{1,c},\ldots,y_{n,c}) = 1_{Y_c}$ and so, $u\gamma_{c+1}(H) \in \mathrm{ker}\xi_c$. Since $\tau(N_c) = \tau_{c+1}(H)/\gamma_{c+1}(H)$, we obtain $\tau(N_c) \subseteq \mathrm{ker}\xi_c$. Let $w = w(y_1,\ldots,y_n) \in \cap_{c\geq 1}\tau_{c+1}(H)$. Then $w \in \tau_{c+1}(H)$ for all c and so, $w(y_{1,c},\ldots,y_{n,c}) = 1_{Y_c}$ for all c. Hence w is a law in $\prod_{c\geq 1} Y_c$. Since $H \leq \prod_{c\geq 1} Y_c$, we have w is a law in H and so, $w = 1_H$. Therefore H is a residually torsion-free nilpotent group.

Finally we show the last part of Theorem B (II). By Theorem A (II), we have $L(c) \cong \mathcal{L}_{\mathbb{Q}}(Y_c)$ for all c via an isomorphism λ_c, say, sending $y_{i,c}$ to $\log h_{i,c}$ for $i = 1,\ldots,n$. As in the proof of Lemma 3.12, there exists a Lie algebra epimorphism $\xi_{\psi_{c,d}}$, with $c \leq d$, from $\mathcal{L}_{\mathbb{Q}}(Y_d)$ onto $\mathcal{L}_{\mathbb{Q}}(Y_c)$ such that $\xi_{\psi_{c,d}}(\log(f\mathrm{ker}\delta_d)) = \log(f\mathrm{ker}\delta_c)$ for all $f \in F_n$. Form the inverse limit of the family $(\mathcal{L}_{\mathbb{Q}}(Y_c), \xi_{\psi_{c,d}})$, $\varprojlim \mathcal{L}_{\mathbb{Q}}(Y_c)$, and define a mapping

$$\lambda : \widehat{L} \to \varprojlim \mathcal{L}_{\mathbb{Q}}(Y_c),$$

as follows:

$$\lambda(u_1 + L', (u_1 + u_2) + \gamma_3(L), \ldots) = (\lambda_1(u_1 + L'), \lambda_2((u_1 + u_2) + \gamma_3(L)), \ldots),$$

where $u_i \in \gamma_i(L)$ for all i. It is easily verified that λ is a Lie algebra monomorphism. Let $v = (v_1, v_2, \ldots) \in \varprojlim \mathcal{L}_{\mathbb{Q}}(Y_c)$. Thus $\xi_{\psi_{c,d}}(\lambda_d(u_d)) = v_c$ for some $u_d \in L(d)$. We claim that

$(u_1, u_2, \ldots) \in \widehat{L}$. Since $\xi_{\psi_{c,d}}\lambda_d = \lambda_c\rho_{c,d}$

$$\begin{tikzcd}
L(d) \arrow[r, "\rho_{c,d}"] \arrow[d, "\lambda_d"'] & L(c) \arrow[d, "\lambda_c"] \\
\mathcal{L}_{\mathbb{Q}}(Y_d) \arrow[r, "\xi_{\psi_{c,d}}"] & \mathcal{L}_{\mathbb{Q}}(Y_c)
\end{tikzcd}$$

for all $c \leq d$ and λ_c is $1-1$, we obtain $\rho_{c,d}(u_d) = u_c$. Hence the Lie algebra monomorphism λ is onto and so, λ is a Lie algebra isomorphism.

For i, $i = 1, \ldots, n$, let

$$t'_i = (\log h_{i,1}, \log h_{i,2}, \ldots, \log h_{i,c}, \ldots) \in \varprojlim \mathcal{L}_{\mathbb{Q}}(Y_c),$$

and let $\Lambda_{\mathbb{Q}}(H)$ be the Lie subalgebra of $\varprojlim(\mathcal{L}_{\mathbb{Q}}(Y_c), \xi_{\psi_{c,d}})$ generated by the set $\{t'_1, \ldots, t'_n\}$. Since L is residually nilpotent, L is embedded into $\widehat{L}$ via a Lie algebra monomorphism λ', say, sending y_i to $(y_i + L', y_i + \gamma_3(L), \ldots)$ for $i = 1, \ldots, n$. Since λ is a Lie algebra isomorphism, we obtain L is isomorphic to $\Lambda_{\mathbb{Q}}(H)$ via $\lambda\lambda'$. Let L_n be the free Lie algebra freely generated by the set $\{\ell_1, \ldots, \ell_n\}$. By the proof of Theorem 4 (for $G_n = H$), we have $\mathcal{L}(H) \cong L_n/\mathrm{ker}\sigma_n$. Furthermore, by applying the proof of Theorem 4 for $\mathcal{L}_{\mathbb{Q}}(Y_c)$ (for all c), we obtain $\Lambda_{\mathbb{Q}}(H) \cong L_n/\mathrm{ker}\sigma'_n$. To prove that $\Lambda_{\mathbb{Q}}(H)$ is a homomorphic image of $\mathcal{L}(H)$, it is enough to show that $\mathrm{ker}\sigma_n \subseteq \mathrm{ker}\sigma'_n$. Since $H/\tau_{c+1}(H)$ is mapped onto Y_c via an epimorphism induced by ξ_c, we have $\mathcal{L}_{\mathbb{Q}}(H/\tau_{c+1}(H))$ is mapped onto $\mathcal{L}_{\mathbb{Q}}(Y_c)$ via a Lie algebra epimorphism sending $\log(y_i\tau_{c+1}(H))$ to $\log h_{i,c}$ for all i. Let $v = v(\ell_1, \ldots, \ell_n) \in \mathrm{ker}\sigma_n$. Then $v(\log(y_1\tau_{c+1}(H)), \ldots, \log(y_n\tau_{c+1}(H))) = 0$ for all c (see the proof of Theorem 4). Hence $v(\log h_{1,c}, \ldots, \log h_{n,c}) = 0$ for all c and so, $v = v(\ell_1, \ldots, \ell_n) \in \mathrm{ker}\sigma'_n$ (by similar arguments as in the proof of Theorem 4). Therefore $\Lambda_{\mathbb{Q}}(H)$ is a homomorphic image of $\mathcal{L}(H)$. □

5 An Example

We shall give an example of a finitely generated Magnus nilpotent group G, not relatively free, such that $\mathcal{L}_{\mathbb{Q}}(G)$ is not isomorphic to $L_{\mathbb{Q}}(G)$ as Lie algebra. We modify (and analyze) an example of a finitely generated nilpotent Lie algebra given in [2, page 210]. Let $\mathcal{L}_4$ be a free Lie algebra of rank 4; freely generated by the set $\{\ell_1, \ldots, \ell_4\}$. Let $L_{4,3} = \mathcal{L}_4/\gamma_4(\mathcal{L}_4)$ and let $x_i = \ell_i + \gamma_4(\mathcal{L}_4)$, $i = 1, \ldots, 4$. Thus $L_{4,3}$ is a free nilpotent Lie algebra of rank 4 and class 3 with a free generating set $\{x_1, \ldots, x_4\}$. Set $v = [x_1, x_2] + [x_3, x_4, x_3]$, and write $L = L_{4,3}/I$,

where I is the ideal of $L_{4,3}$ generated by v. It is easily verified that every element of I has the form

$$a([x_1, x_2] + [x_3, x_4, x_3]) + \sum_{i=1}^{4} a_i[x_1, x_2, x_i],$$

where $a, a_i \in \mathbb{Q}$, $i = 1, \ldots, 4$. For $i = 1, \ldots, 4$, let $y_i = x_i + I$. Then $[y_1, y_2] = -[y_3, y_4, y_3] \in \gamma_3(L)$ and $[y_1, y_2, y_i] = 0$ with $i = 1, \ldots, 4$. Since I is a proper subset of $L'_{4,3}$, we obtain L is not abelian, and the set $\{y_i + L'; i = 1, \ldots, 4\}$ is a $\mathbb{Q}$-basis of L/L'. Since L is finitely generated nilpotent Lie algebra and $\dim(L/L') = 4$, we obtain L is generated by the set $\{y_1, \ldots, y_4\}$. Suppose that L is relatively free. Since $\dim(L/L') = 4$ and L is nilpotent, it is easily verified that the set $\{y_1, \ldots, y_4\}$ freely generates L. Then $[y_1, y_2] = [y_3, y_4, y_3] = 0$ and so, $[x_1, x_2] \in I$ which is not valid. Thus L is not relatively free.

Give on L the structure of a group, denoted R, by means of the BCH formula. That is, for $x, y \in L$,

$$x \circ y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, y, y] - \frac{1}{12}[x, y, x]. \tag{17}$$

Recall that 0 is the unit of R and $-x$ is the inverse of x with respect to the group operation $\circ$. Let H be the subgroup of R generated by the set $\{y_i; i = 1, \ldots, 4\}$. For $u, v \in H$,

$$(u, v) = [u, v] + \frac{1}{2}[u, v, u] + \frac{1}{2}[u, v, v]. \tag{18}$$

Thus

$$(y_1, y_2) = (y_3, y_4, y_3)^{-1}.$$

By Lemma 2.1, $\tau_2(H) = H'$, and so, H/H' is free abelian of rank 4. By (17), for $\alpha \in \mathbb{Z}$ and $u, v \in H$, $(u, v)^\alpha = \alpha(u, v)$. By (18), we obtain

$$(u, v)^\alpha = \alpha[u, v] + \frac{\alpha}{2}[u, v, v] + \frac{\alpha}{2}[u, v, u]. \tag{19}$$

Suppose that

$$\prod_{i=1}^{2}(y_3, y_i)^{\alpha_i} \prod_{j=1}^{3}(y_4, y_j)^{\beta_j} \in \gamma_3(H),$$

where $\alpha_i, \beta_j \in \mathbb{Z}$, $i = 1, 2$ and $j = 1, 2, 3$. Since $\gamma_3(H)$ is generated by the elements of the form (y_i, y_j, y_k) with $i, j, k \in \{1, \ldots, 4\}$, we obtain from (19) and the form of elements of I that

$$a[x_1, x_2] + \sum_{i=1}^{2} \alpha_i[x_3, x_i] + \sum_{j=1}^{3} \beta_j[x_4, x_j] \in \gamma_3(L_{4,4}),$$

for $a \in \mathbb{Q}$. Since $L_{4,3}$ is a free nilpotent Lie algebra, we have $\alpha_i = 0$ ($i = 1, 2$) and $\beta_j = 0$ ($j = 1, 2, 3$). Therefore $H'/\gamma_3(H)$ is torsion-free. Finally, it is easily shown that $\gamma_3(H)$ is torsion-free. Thus H is a Magnus group. Suppose to a contrary that H is relatively free. Then, by Theorem A (I), $\mathcal{L}_{\mathbb{Q}}(H)$ is relatively free. Since both R and $\exp \mathcal{L}_{\mathbb{Q}}(H)$ are Mal'cev completions of H, we obtain $R \cong \exp \mathcal{L}(H)$ as groups and so, $L \cong \mathcal{L}_{\mathbb{Q}}(H)$ as Lie algebras. That is, L is relatively free which a contradiction, and so, H is not relatively free. Suppose that there exists a Lie algebra isomorphism ζ from $L_{\mathbb{Q}}(H)$ into L. By Lemma 3.2, $L_{\mathbb{Q}}(H)$ is generated by the set $\{y_1H', \ldots, y_4H'\}$. Write $\zeta(y_iH') = z_i$, $i = 1, \ldots, 4$. Since

$$[y_1H', y_2H'] = (y_1, y_2)\gamma_3(H) = 0 \text{ in } L_{\mathbb{Q}}(H),$$

we have $[z_1, z_2] = 0$ in L. For $i = 1, 2$,

$$z_i = \sum_{j=1}^{4} \alpha_{ji} y_j + \sum_{\substack{\kappa \neq 2 \\ 1 \leq \ell < \kappa \leq 4}} \beta_{\kappa \ell i}[y_\kappa, y_\ell] + v_i,$$

where $v_i \in \gamma_3(L)$. Since $\{z_1 + L', z_2 + L'\}$ is linearly independent, we obtain $\alpha_{\mu 1}\alpha_{\nu 2} - \alpha_{\nu 1}\alpha_{\mu 2} \neq 0$ for some $\mu, \nu \in \{1, \ldots, 4\}$ with $\mu \neq \nu$. Since $[y_1, y_2] + [y_3, y_4, y_3] = 0$, and working with a basis of L consisting of "basic commutators", we obtain the following equations

$$\begin{aligned}
\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} &= -\alpha_{32}\beta_{431} + \alpha_{31}\beta_{432} \\
\alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31} &= 0 \\
\alpha_{11}\alpha_{42} - \alpha_{12}\alpha_{41} &= 0 \\
\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31} &= 0 \\
\alpha_{21}\alpha_{42} - \alpha_{22}\alpha_{41} &= 0 \\
\alpha_{31}\alpha_{42} - \alpha_{32}\alpha_{41} &= 0 \\
\alpha_{41}\beta_{432} - \alpha_{42}\beta_{431} &= 0
\end{aligned}$$

Thus $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} \neq 0$. Therefore $\alpha_{11}\alpha_{22} \neq 0$ or $\alpha_{12}\alpha_{21} \neq 0$. Suppose that $\alpha_{11}\alpha_{22} \neq 0$. If $\alpha_{12}\alpha_{31} \neq 0$ then $\alpha_{11}\alpha_{32} \neq 0$ and so $\frac{\alpha_{11}}{\alpha_{12}} = \frac{\alpha_{31}}{\alpha_{32}}$. Furthermore $\frac{\alpha_{21}}{\alpha_{22}} = \frac{\alpha_{31}}{\alpha_{32}}$. Hence $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 0$ which is a contradiction. If $\alpha_{12}\alpha_{31} = 0$ then $\alpha_{12} = 0$ or $\alpha_{31} = 0$. Since $\alpha_{11} \neq 0$ we get $\alpha_{32} = 0$. Then $\alpha_{22}\alpha_{31} = 0$ and so $\alpha_{13} = \alpha_{32} = 0$ which is a contradiction. Similar arguments may be applied if $\alpha_{12}\alpha_{21} \neq 0$. Therefore we obtain L is not isomorphic to $L_{\mathbb{Q}}(H)$ as Lie algebras.

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